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*MUTUALLY COUNTERACTING PINHOLE PROBES**

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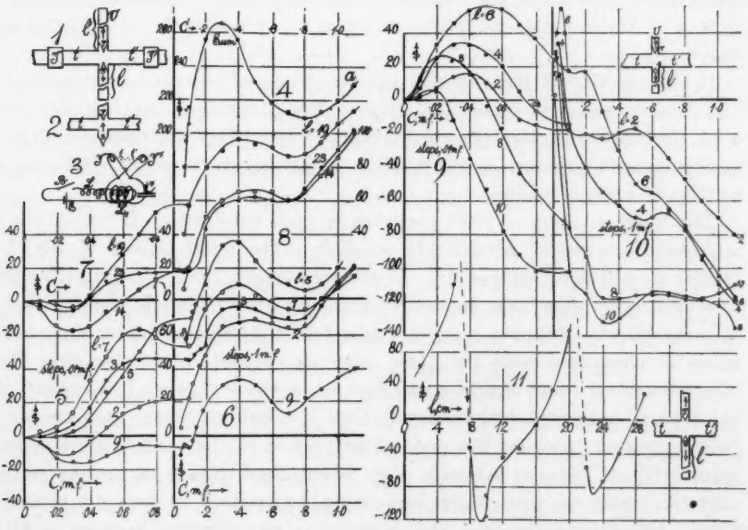
1. *Apparatus.*—In figure 2 (notation as in Fig. 1), where tt' is the pipe connecting the telephones at T and T', the pinholes r and s are set to cooperate with each other. The stream lines run from the apex to the base



of each cone (so called[1] and so drawn) and beyond r at the U-gauge; there is thus evidence of pressure. The mean density of the node between r and s is an excess. Only the points of the pinhole cones are effective. They may be pricked in thin metal foil and will act positively or negatively, according as the puncture is differently shaped. Size of hole and slope of walls are important, but the volume of the region in which the action is completed is astonishingly small. At a venture one may suppose

that within the apex of the hollow cone there is marked vorticity and that velocity here is changed to pressure, but the adequate reason for the reversibility of pinhole probes is yet to be shown.

In figure 3 the adjustment for electric oscillations are diagrammatically given (e, electromotive force—3 cells with resistance—B, platinum spring break of low frequency, P, primaries in series, L actuating the spring break, L_1 secondary with capacity C and telephones T, T' connected by the pipe tt').

Figure 4, curve a, is an example of the fringe displacement s of the U-gauge interferometer, for the summational adjustment of pinholes, when the capacity in the secondary is varied in steps of 0.1 microfarad. There is a maximum at $C = 0.3$ microfarad about, corresponding to the oscillation pitch near d'', and a minumum at $C = 0.8$ microfarad corresponding to f', at which the pipe makes its nearest approach to silence, though there is abundance of multiresonance left here. Nevertheless the crest for $C = 0.3$ and the trough for $C = 0.8$ must dominate the whole of the subsequent experiments, since all are derived from the acoustic oscillation in the pipe tt'. With better adjustment the crest could easily be doubled in height s.

It follows that if the pin-hole probes, r, s, are adjusted as in figure 1 (T, T' telephones in phase, tt' acoustic pipe, interferometer U-gauge beyond U) they will counteract each other. The effect at U will be differential, either a pressure or a dilatation, according to the relative efficiency of the two pinhole probes.

The inner of these r is left constant in quill tube length ($l' = 10$ cm.) and its effect will be persistently positive. The outer pinhole s (Fig. 1) carries a quill tube extension, l, of variable length. Such a quill tube (l) as shown heretofore[2] is a musical instrument with a pinhole embouchure; but the effects obtained by the method of the present paper are enormous as compared with the older evidence. It is presumable that the effectiveness of s and also of r will depend on the frequency of the note of the pipe tt' primarily (remembering that a telephone plate overtone may be in question) and on the natural frequency of the pinhole quill tube, secondarily. The experiments with counteracting pinhole probes were, therefore, made by successively increasing the length, l, of the outer pinhole tube. For each value of l, the capacity C was changed in steps of 0.01 microfarad from 0 to 0.1 m.f. (Figs. 5, 7, 9) and in steps of 0.1 m.f. from 0 to 1 microfarad (Figs. 6, 8, 10), to bring out the nature of the phenomena.

2. *Positive Fringe Displacements, s, Prevailing.*—Figure 5 shows that the graphs $l = 2, 3, 5, 7$ cm., though different in character, follow each other in regular succession for values of C below 0.06 microfarad; but between $l = 7$ and $l = 9$ cm. there is instability, so that the former graph drops

over a relatively large range in s, from positive to negative values. There is a definite crest for $l = 7$. The other curves run into a plateau which in figure 6 appears merely as double inflection. Consequently, in the latter case, the interval 0.1 to 0.2 microfarad was also often worked out in steps of 0.01, but no ignored crests were detected. In figures 7, 8, the lengths, l, are taken larger to show that the periodicity of graphs has not ceased.

Figures 6 and 8 clearly indicate that all the graphs are dominated by the summational curve a, figure 4 (pinholes in series), at least insofar as the position of minima is concerned. The maxima, however, are shifted to the right, i.e., to lower frequency, as if the pipe note had flattened.

3. *Negative Fringe Displacements, s, in Excess.*—In figures 5–8 the inner pinhole prevails; at least in the region from $C = 0.1$ to $C = 1.0$ m.f., the curves tend to be strongly positive throughout. Search was, therefore, made for an outer pinhole which would have the same negative excess qualities if placed in the position s of figure 1. After many trials one only was found. The fringe displacements with this probe were much more sluggish than in the preceding experiments, indicating a pinhole of much finer bore.

In figure 9, for steps of 0.01 m.f., the curves $l = 2, 4, 6$ cm. are a progression, with the crests moving to the right. After this there is an instability with a drop in s values from $l = 6$ to 8 and 10 cm., the crests tending to the left. In the former cases, the inner pinhole is still dominant; but it ceases to be so in the latter ($l = 8$, 10 cm.). One may notice, in general, that positive tendencies in s here often replace negative tendencies in figures 5–7.

In figure 10 for $C = 0.1$ to 1.1 m.f., the resemblance to the summational curve (supplied in Fig. 4 at a) is practically eliminated. The strong summational crest at $C = 0.3$ microfarad is obscurely replaced by troughs, if at all. The trough at $C = 0.8$ has no correspondence; but there is in this region a shift of maxima. The intense negative characteristics of this reversed pinhole $l = 8$, 10 cm. are remarkable.

Unfortunately in endeavoring to improve this pinhole No. 11 by washing it, the negative character completely vanished. It must have been due, therefore, to something like atmospheric accretions accumulated in years by which the pinhole was incidentally constricted in such a way to give it its negative quality.

4. *Summary.*—The fringe displacement, s, at $C = 0.1$ m.f., may be taken for the purpose of discrimination, at least at the outset. These data, obtained in a later and more extensive survey, may be tabulated as follows:

C	0.1 microfarad; pinhole as in §2											
l	2	4	6	8	10	12	14	16	18	20	22	29 cm.
s	63	100	140	−33	−90 / −115	−45	−35	−7	35	105	−57	30

They are constructed in figure 11, which may be regarded as a summary of figures 5–8. The detailed account which the pinhole probe gives of the nature of the vibration in quill tubes is amazing. Figure 11 admits of a straightforward interpretation. When the quill tube of the outer pinhole is elongated, the vibration for a given harmonic within ceases more and more to eventual silence. Hence the inner or positive pinhole is continually more effective to a positive maximum. After this the outer pinhole vibration drops to the next harmonic of lower frequency. Corresponding to the better fit of the longer tube, the outer vibration is then suddenly intensified, and is in excess of the inner pinhole vibration. Negative values of s, therefore, supervene to be gradually diminished, in turn, on further elongation of the outer tube. The positive increments of the graph, figure 11, are thus continuous and the changes gradual; the negative increments sudden and the action impulsive. This figure and others of a similar nature show that to pass from cusp to cusp an elongation of quill tube of about 14 cm. was needed.

The above work which encounters the superposition, more or less, of four harmonic graphs, those of the electric oscillation and the pipe oscillation and those of the two pinhole tubes individually is naturally destined to run into complications. It was, therefore, thought best to avoid speculation. The abscissas $C = 1/L(2\pi n)^2 = \lambda^2/L(2\pi v)^2$ increase with the squares of period or wave-length. Increasing l of the outer pinhole increases λ. If a C is found to fit, it does not generally fit the fixed l' of the inner pinhole, nor the pipe tt'. There is a further outstanding factor in the fit of the bore, etc., of the pinhole to the note. Finally overtones in the telephone plate will be changed with small changes in the pitch of the electric spring break.

If we turn back to figure 1 of an earlier paper[3] giving the survey in pitch of the tube tt', we are struck by the extremely sharp cusps at the harmonics, alternating with intervals of complete silence. This implies the use of a periodic break of variable frequency. In contrast with this succession of cusps such a curve as figure 4, obtained with the spring break of constant low pitch, is illuminating. The tube tt' is never silent. It follows that high harmonics fitting the pipe tt' must be shaken out of the telephone plates, throughout and in succession, each in turn accentuated by the pipe. The drops of curves observed in figures 2 and 10, however, is probably incident in the quill tube itself additionally; but the absence of silence in all of them is to be referred to the pipe tt'. It follows that if sharp cusps and precise values of L or C are to be obtained, the motor periodic break will have to be used in connection with tt'.

Finally in the association of a fine outer (reversed) and coarser inner pinhole and at the lower frequencies ($C > 0.1$ m.f.), the fine pinhole is more efficient and the resulting graphs in s, negative.

* Advance Note from a Report to the Carnegie Inst., of Washington, D. C.

[1] With more sensitive apparatus, I have since obtained similar acoustic pressures, with clear quill tubes free from pinholes.

[2] Carnegie Publications, Washington, No. 310, 1921, § 24, et seq.

[3] These PROCEEDINGS, **11**, 1925, p. 582, for instance.

TRANSITION PROBABILITIES AND PRINCIPAL QUANTUM NUMBERS

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On first consideration it would appear that a simple means of confirming or correcting an assignment of quantum numbers to the terms of a spectrum is afforded by a comparison of transition probabilities calculated on the basis of the correspondence principle with values obtained experimentally from the intensities of the lines, their dispersion or similar properties. Any such information as to the principal quantum numbers in series spectra would be particularly valuable, for the assignment of these numbers as proposed by Bohr involves a consideration of the arrangement of electron orbits in the atom and its relation to the periodic properties of the elements. However, on closer examination it appears that if the calculation is made on the basis of the ordinary conception of the correspondence principle the transition probabilities in series spectra will be to a first approximation independent of the principal quantum numbers assigned to the terms, in spite of the fact that the order of the corresponding harmonic is determined by the change in these numbers during a transition. This approximate independence follows as a result of the general assumptions as to the nature of the motion of the series electron which have been used for an interpretation of the main features of series spectra. It is true that with an exact knowledge of the motion of the series electron a dependence of the transition probabilities on the principal quantum numbers would result, but from such a detailed knowledge it would be possible to determine the quantum numbers directly.

Thus from the standpoint of the quantum theory in the form on which calculations have so far been based it does not appear that the discussion of transition probabilities affords a practical method for checking the principal quantum numbers. In particular the author is unable to agree with the tentative conclusions drawn by W. Kuhn[1] from his measurements of the magnetic rotation of the plane of polarization in thallium vapor as to

the necessity of modifying Bohr's original assignment of principal quantum numbers in this element.

The reasons for the above statements will appear more clearly if we consider the way in which it is possible to estimate the corresponding amplitudes, from which the transition probabilities may be calculated. A formal method of making such an estimate, following as closely as possible the theories of Bohr and Kramers, has been outlined in two papers by the author.[2,3] There it was considered that to a first approximation the motion of the series electron is a central motion and hence doubly periodic, with frequency ω for the radial motion and σ for the superposed rotation. The Fourier series representing such a motion contains frequencies $|\tau\omega + \sigma|$ and $|\tau\omega - \sigma|$ where τ is the order of the harmonic, and in rectangular coördinates it may be written

$$x + iy = \sum_{-\infty}^{+\infty}{}_{\tau} C_\tau e^{2\pi i(\tau\omega+\sigma)t}.$$

As to the amplitudes C_τ it has been shown that to within an approximation which is essentially that with which the energies may be represented by a formula of the Rydberg type they are given by the expression

$$C_\tau = \frac{a}{\pi} \sin \rho\pi \sum_{-\infty}^{+\infty} b_m J_m(\rho\epsilon). \qquad (1)$$

The approximation applies to orbits which may be thought of as composed of outer segments in which the motion is nearly Keplerian joined by inner segments in which the deviations from Keplerian motion may be large, but which are traversed in a time which is short compared to the period of the radial motion. In equation (1) a and ϵ are, respectively, the semi-major axis and eccentricity of the Keplerian ellipses of which the outer segments form a part, while b_m is an algebraic factor involving only ϵ and ρ, and J_m is a Bessel function of order m. The order τ of the harmonic enters only through ρ which has been introduced as an abbreviation for the quantity $\tau + \sigma/\omega$ and may be thought of as an "effective order" of the harmonic. Since, then

$$\rho = \tau + \sigma/\omega. \qquad (2)$$

it is readily seen that we may write for the frequency $\omega_0 = \tau\omega \pm \sigma$ of any harmonic,

$$\omega_0 = \rho\omega.$$

For the purpose of calculating the transition probabilities one considers the spontaneous radiation from a given stationary state as equivalent to that of a set of classical oscillators with electric moments

$$eA_k \cos 2\pi\nu_k t$$

where the ν_k are the frequencies associated by the quantum relation with the possible transitions to states of lower energy, and A_k is connected with Einstein's probability coefficient by the equation,

$$A'_{r,}h\nu_k = \frac{32\pi^4 e^2}{3c^3}\nu_k^4 A_k^2.$$

Since ν_k may be represented as an appropriate mean value of a "corresponding" frequency $|\tau\omega \pm \sigma|$ in the motion of the series electron namely that for which $\Delta n = \tau$, $\Delta k = \pm 1$, where n and k are the principal and subordinate quantum numbers, it may be assumed that A_k can also be represented as some average value of the amplitude C_τ of this corresponding harmonic. In many cases this mean value is probably given with sufficient accuracy by Eq. (1) with substitution of mean values of the parameters a and ϵ and of the effective order ρ.

Reasonable mean values of these quantities may be found in terms of the energies or term values of the stationary states involved, or what amounts to the same thing, in terms of their effective quantum numbers as defined by the equation

$$E = -Th = -\frac{Rh}{n^{*2}} \qquad (3)$$

where E is the energy of the series electron, T the term value and R the Rydberg constant. We may, for instance, use as the effective mean value $\overline{n^*}$ the simple arithmetical mean of n^* in the stationary states and similarly for $\overline{k}$, in which case $\bar{a}$ and $\bar{\epsilon}$ are given by the well-known relations

$$\bar{a} = \frac{e^2}{2Rh}\overline{n^{*2}}, \quad \bar{\epsilon} = \sqrt{1 - \frac{\overline{k^2}}{\overline{n^{*2}}}}.$$

To obtain the mean value $\bar{\rho}$ we must estimate the mean value of σ/ω, and it appears possible to do this formally in terms of n^*, by means of the relation which exists between the frequencies in a multiply periodic motion and the quantum numbers. In fact, as previously shown, the ratio σ/ω is equal to the negative of the derivative of the "quantum defect" q with respect to k, i.e., $\sigma/\omega = -\frac{dq}{dk}$ where q is defined by the equation

$$E = -\frac{Rh}{(n-q)^2}. \qquad (4)$$

As Bohr has shown[4] the quantum defect, q, is, on the basis of just the assumptions used in deriving (1), independent of n and a function of k alone, and the general approximate validity of (4), which is the Rydberg formula, is strong evidence for the correctness of these assumptions. For an aver-

age of $\frac{dq}{dk}$ it seems reasonable to take $\frac{\Delta q}{\Delta k}$ where Δ denotes the finite difference between initial and final states, and by virtue of the fact that k changes always by one unit we have

$$\overline{\sigma/\omega} = -\Delta q. \tag{5}$$

By comparison of (3) and (4) we see that $n^* = n - q$ and from (2) and (5) it follows that we may write

$$\bar{\rho} = \tau + \sigma/\omega = \Delta n - \Delta q = \Delta n^*.$$

The quantities $\bar{a}$, $\bar{\epsilon}$ and $\bar{\rho}$ and hence A_k are thus given in terms of the energies in the stationary states alone and the transition probabilities as calculated in this way will not be altered by changing arbitrarily the principal quantum numbers of the terms. This may perhaps be more simply demonstrated by noting that if any quantum number is changed by a given number of units σ/ω and τ will be changed by the same number of units but in opposite senses, so that $\bar{\rho}$ remains unaltered.

Besides showing clearly that the transition probabilities may be calculated approximately from the term values alone, the procedure outlined above is very convenient for the numerical calculation of the corresponding amplitudes. On the other hand no strict justification can be given for the use of the method of averaging employed, although some support for it may be found in the fact that if it is used for a calculation of the frequencies $\left(\text{by means of the relation } \nu = \bar{\omega}_0 = \bar{\rho}\,\bar{\omega} = \Delta n^* \frac{2R}{\bar{n}^{*3}}\right)$ an approximate agreement with the experimental frequencies is obtained.

The considerations of this note have been based entirely on what may already be called the "classical" form of the quantum theory. The recent reformulation of the theory due to Heisenberg[5] and Born and Jordan[6] permits a more direct calculation of the transition probabilities, although as yet the calculation cannot be carried through for the central motions here considered. Since, however, there is approximate agreement between the results of the two theories in many cases, one may hazard the guess that in the newer form of the theory the quantity analogous to $\tau + \sigma/\omega$ will appear as a whole, not separable into τ and σ/ω and will determine the transition probabilities, giving the same type of independence of the principal quantum numbers as deduced here from the older form of the theory.

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[1] W. Kuhn, *Naturwissenschaften*, **13**, 724 (1925).

[2] F. C. Hoyt, *Physic Rev.*, **25**, 174 (1925).

[3] F. C. Hoyt, *Physic Rev.*, **26,** 749 (1925).
[4] See M. Born, "Vorlesungen über Atommechanik," Berlin, 1925, p. 184.
[5] W. Heisenberg, *Zeit. Physik*, **33,** 879 (1925).
[6] M. Born and P. Jordan, *Ibid.*, **35,** 858 (1925).

ABSORPTION AND RESONANCE RADIATION IN EXCITED HELIUM AND THE STRUCTURE OF THE 3889 LINE

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In a recent issue of *Nature*[1] a brief letter was published on the results of some experiments on the optical properties of excited helium. A detailed account of the experimental work and plates representing the results obtained are being sent to the *Philosophical Magazine* for publication. The following is a summary of those results, with the addition of some discussion of their significance.

The method used differed from the usual photometric method in that the light, after passing through the absorbing column, was resolved by means of an echelon grating and examined for a weakening of the central portion of the line, which would be affected by the absorption. It was found possible to completely reverse several lines of the helium spectrum by an absorbing column 80 cm. in length. The lines reversed included not only the principal series of both the singlet and doublet systems, but also the diffuse subordinate series of both systems, not, however, the lines of the sharp subordinate series. Table 1 gives a list of the lines which were reversed.

TABLE 1

HELIUM LINES REVERSED BY ABSORPTION IN A COLUMN OF EXCITED HELIUM OF LENGTH 80 CM. PRESSURE OF GAS 4.0 MM.

WAVE-LENGTH	LINE	WAVE-LENGTH	LINE
3889 Å	1σ–2π	5016 Å	$1S$–$2P$
3187	1σ–3π	3964	$1S$–$3P$
5876	1π–2δ	3614	$1S$–$4P$
4471	1π–3δ	6678	$1P$–$2D$
4026	1π–4δ	4922	$1P$–$3D$
3819	1π–5δ		

In agreement with the results of other observers, it was found that the presence of small amounts of hydrogen in the discharge greatly reduced

the absorption. This is probably due to a decrease in the life of the excited atoms produced by the presence of the hydrogen.

At high pressures the doublet system lines were more strongly absorbed than were the singlet system lines, while at low pressures the reverse was found to be the case. This is in agreement with the variation of the relative intensity of the two systems with pressure in the emission spectrum of helium. The doublets are relatively intense at high pressure and the singlet lines at low pressure. This shows definitely that the excitation of helium by electron impacts is predominantly to the 1σ state at higher pressures while low pressures the excitation to the $1S$ state is more important.

For a given series the absorption was found to fall off rapidly as the higher terms of the series were considered. In the case of the 1σ–$m\pi$ series only the first three terms have been studied and all show strong absorption. In the case of the 1π–$m\delta$ series absorption could be detected only to the fourth term. With the $1S$–mP series absorption was found to the fourth term and in the case of the $1P$–mD series only to the second term, and then only with strong excitation in the absorbing column.

Discussion.—In considering the emission spectrum of a gas, the decrease of intensity of the lines of the series as the higher terms are considered, may be due to two causes; (*a*) the number of atoms in the states of higher energy which may move to a state of lower energy with the emission of radiation. Then as the higher terms are considered the decrease of intensity may be attributed to a decrease in the number of atoms in the states of high energy.

(*b*) The probability of the transition considered occurring. In dealing with the absorption of a given series, only (*b*) can influence the results as all the lines of a given series are due to transitions of atoms from the same initial state. Thus the falling off of the absorption with the higher terms of the series, must be due to a decrease in the probability of a transition to the states of higher energy.

The importance of the probability factor is still further emphasized in the result that the sharp subordinate series 1π–$m\sigma$ and $1P$–mS are not absorbed, while the diffuse series 1π–$m\delta$ and $1P$–mD are absorbed. Both series have the same initial states so the difference in the absorption can be due only to the difference in the probability of the transitions.

A study of the absorption of the different spectral lines in a gaseous discharge, under varying conditions, should throw some light on the complicated phenomena. In most considerations of gaseous discharge, the possible influence of excited atoms is neglected. In considering the striated discharge in low-voltage discharge tubes, it was pointed out that impurities may affect the length of life of the excited atoms.[2] This appears to be confirmed to some extent by the results of the present work.

In addition to the preceding, a successful attempt has been made to photograph resonance radiation in the weakly excited helium. Care was taken to eliminate scattered light and the excited helium was strongly radiated with light from a second tube filled with helium. After proving that the scattered light from the second tube was inappreciable, it was found that the intensity of the 3889 Å line was greatly increased by the action of the radiation from the second tube. This must be due to resonance of the excited gas, produced by light of the same wave-length from the exciting tube. Paschen observed the same phenomenon for the first line of this series, but this appears to be the first photographic record of combined electrical excitation and resonance.

The structure of the 3889 line, studied by means of a quartz Lummer Gehrcke plate with the discharge tube cooled in liquid air, has been found to consist of at least two components. A weak component was found 0.044 Å to the short wave-length side of the main component. Photographs representing the structure of this line as well as the other results obtained are being published with the experimental work.

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[1] McCurdy, *Nature*, p. 122, January 23, 1926.

[2] McCurdy, *Phil. Mag.*, **48**, p. 898, 1924.

THERMODYNAMIC DERIVATION OF A BLACK BODY RADIATION ISOTHERM

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With the aid of thermodynamic considerations it is intended to show that Wien's Displacement Law and an additional assumption are sufficient to define the equilibrium state of a system. It is further shown, that this system then will obey Stephan-Boltzmann's Law and that the previous additional assumption is the thermodynamic explanation of the constancy of h—Planck's constant.

Let us consider a process

$$A \longrightarrow B - Q \qquad (1)$$

connected with the absorption of heat Q. We shall imagine it to be an ideal process of evaporation: i.e., the solid and the gas will be immaterial, therefore the volume of the solid will be negligible as compared with the volume of the gas; the gas will have a definite equilibrium pressure for

every given temperature. In a mixture it will strictly obey Dalton's Law of partial pressures, that is, its partial pressure will be independent of any other gas present. We will not introduce the atomistic concept and will therefore deal with the heat of reaction Q given in ergs per unit volume.

To a system of this sort we can apply strictly the Second Law of Thermodynamics in the form of van't Hoff's reaction isochore:

$$\frac{dlnp}{dT} = \frac{Q}{nT^2} \tag{2}$$

where n stands for the gas constant per unit volume, p is the pressure and T the absolute temperature.

Let us further consider an infinite number of other processes of the same form

$$A \longrightarrow B_\lambda - Q_\lambda, \text{ etc.} \tag{1a}$$

They will differ between themselves by the product of the process (B_λ), by the heat effect connected with it (Q_λ) and by the equilibrium pressures (p_λ). As we are going to deal with the properties of every single process, we shall identify them by giving them numbers (λ); to have the condition of infiniteness every rational positive number from $0 \div \infty_*$ will have its significance by representing a special process. We shall further assume that the properties of these processes change continually with their numbers, and therefore any property of two components will approach identity when the difference in number $\Delta\lambda$ will approach $d\lambda$ and 0. To every independent component of our system we shall apply the van't Hoff isochore.

Having defined so far our system we shall impose a restriction on it: namely we shall assume that Q is independent of temperature (1st assumption). This permits us to integrate (2) and we obtain

$$lnp = -\frac{Q}{nT} + \text{const. (with resp. to } T). \tag{3}$$

It remains to specify the two values Q and "const." to have our system defined. This can be illustrated and proved in the following way: If we draw lnp against $1/T$, we obtain a straight line with the slope $-Q/n$. As long as we have not accepted any regularities for the individual processes of our system, any set of straight lines will satisfy the conditions we have so far imposed. But with respect to functional dependence our system becomes quite definite with the following assumptions:

The $lnp-1/T$ curves intersect each other and the contour line of their intersection has the form

$$p_m = \beta T^5 + \gamma \text{ (2nd assumption)} \qquad (4)$$

i.e., there will correspond to every temperature a certain $p_{max.}$ given by this equation. (Fig. 1.)

We further assume that if we take a definite T, that then the tangent to the corresponding p_m is the lnp straight line for the component of our system which we can find from

$$\lambda_m = \frac{\alpha}{T} \text{ (3rd assumption)} \qquad (5)$$

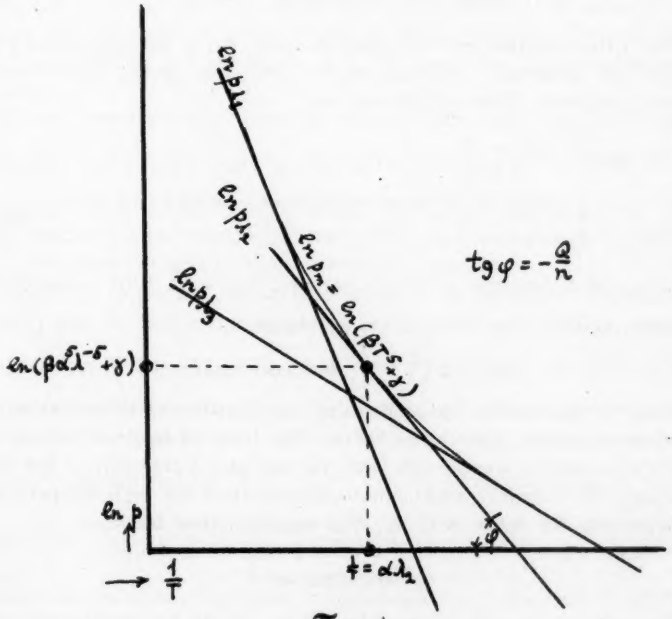

Fig. 1.

The constants α, β and γ are independent from λ and T. It is clear that under these conditions $\frac{Q}{n}$ can be easily found:

$$lnp_m = ln(\beta T^5 + \gamma)$$

$$\frac{dlnp_m}{d(\frac{1}{T})} = -\frac{5\beta T^6}{\beta T^5 + \gamma} = -\frac{5\beta T}{\beta + \gamma T^{-5}} = -\frac{Q}{n}$$

according to the 3rd assumption for $Q = Q_\lambda$, $T = \frac{\alpha}{\lambda_m}$, therefore

$$Q_\lambda = \frac{5\alpha\beta}{\beta + \alpha^{-5}\gamma\lambda^5} \cdot \frac{n}{\lambda}. \tag{6}$$

From the conditions given above it is also evident that the equation for the *lnp* of an individual process will be

$$ln p_\lambda = ln(\alpha^5\beta\lambda^{-5} + \gamma) + \frac{5\beta}{\beta + \gamma\alpha^{-5}\lambda^5} - \frac{5\alpha\beta}{\beta + \gamma\alpha^{-5}\lambda^5} \cdot \frac{1}{\lambda T}$$

and

$$p_\lambda = (\alpha^5\beta\lambda^{-5} + \gamma)\, e^{\frac{5\beta}{\beta+\gamma\alpha^{-5}\lambda^5}} \cdot e^{-\frac{5\alpha\beta}{\beta+\gamma\alpha^{-5}\lambda^5} \cdot \frac{1}{T\lambda}}. \tag{8}$$

This is the equation for the vapor pressure (p_λ) of the individual component (*the isotherm*). We can easily verify the proper imposition of our assumptions. Differentiation gives

$$\frac{dp\lambda}{d\lambda} = 5\beta e^{\frac{5\beta(\lambda T-\alpha)}{(\beta+\gamma\alpha^{-5}\lambda^5)\lambda T}} \left[\frac{(\alpha^5\lambda^{-5}\beta+\gamma)(\beta+\gamma\alpha^{-5}\lambda^5)\lambda T-(\lambda T-\alpha)(\beta+6\gamma\alpha^{-5}\lambda^5)}{T(\beta + \gamma\alpha^{-5}\lambda^5)^2\lambda^2} - \alpha^5\lambda^{-6}\right]$$

this expression becomes evidently zero for $\lambda_m = \frac{\alpha}{T}$ (3rd assumption). Substituting this value of λ_m in (8) we obtain

$$p_m = \beta T^5 + \gamma \text{ (2nd assumption).}$$

Before we proceed to the analysis of our distribution curve (8) an approximation can be introduced. From the form of equation (4) we see that if γ is small or comparable with β it will play a rôle only at low temperatures, we therefore approximate our equation for high temperatures by neglecting the terms with γ. Our equation then becomes

$$p_\lambda = e^5\alpha^5\beta\lambda^{-5}e^{-\frac{5\alpha}{\lambda T}}. \tag{9}$$

This equation is easily integrable and we obtain for the integral pressure of our system (P) at a definite temperature

$$P = \int p_\lambda d\lambda = e^5\alpha^5\beta e^{-\frac{5\alpha}{\lambda T}} \left(\frac{T}{5\alpha\lambda^3} + \frac{3T^2}{5^2\lambda^2\alpha^2} + \frac{6T^3}{5^3\alpha^3\lambda} + \frac{6T^4}{5^4\alpha^4}\right)$$

and

$$\underset{\lambda=\infty}{P} = \int_0^\infty p_\lambda d\lambda = \frac{6}{5^4} e^5\alpha\beta T^4 = \sigma T^4$$

that is, the total pressure of our system increases with the fourth power of the absolute temperature.

Summarizing the results of our consideration we may state that if we are given a system of an infinite number of independent processes of the form $A \longrightarrow B - Q$, connected by three conditions, (1) constancy of the heat effect of the individual process with respect to temperature, (2) increases of the maximum partial pressure of the system as the 5th power of the absolute temperature, and (3) the restriction that this maximum pressure corresponds to the component reaction λ given by the equation $\lambda_m = \frac{\alpha}{T}$, we then arrive at a functionally definite single system with an energy distribution given by equation (8) and with the total energy varying as the 4th power of the absolute temperature (11). In other words every system obeying conditions 1–3, will also have to follow equations (8) and (11).

We only need to realize that the energy content of a black body differs from radiation pressure only by constants ($E = \frac{3}{4}cp$, given by the electromagnetic theory) to see that conditions 2 and 3 are conclusions from Wien's Displacement Law,[1] that equation (8) obeys Wien's Displacement Law and that equation (11) is Stephan-Boltzmann's Law.

It remains to see how far our assumption (1) is justified. We have seen already that at high temperatures we can neglect terms with γ, equation (6) therefore becomes $Q_\lambda = \frac{5\alpha n}{\lambda}$ and $\frac{Q}{nT} = \frac{5\alpha}{\lambda T}$ is the value which appears in the exponential of (9). But (9) differs from Planck's equation only by the correction factor (-1), which disappears for small values of λ. Planck's exponential factor is $\frac{hc}{\kappa\lambda T}$. We therefore can compare the two quantities. It is evident that 5α should correspond to $\frac{hc}{\kappa}$. In fact experimental determinations[2] give $\frac{hc}{\kappa}$ to be equal to 4.965α, that is, within 0.7 per cent of the predicted approximated value. The importance of a value $h\nu$ and the constancy of h is herewith reduced to the constancy of Q with respect to temperature and its practically exact inverse proportionality to λ.

Completing the comparison of (8) with Planck's formula we see that for small values of λ both go over into Wien's formula. At low temperatures mainly for large λ a correction factor appears, it is -1 in Planck's equation and γ in the proposed one, both decrease the slope of the isotherm as compared with the slope of Wien's Radiation Formula.

Further development of the proposed formula is reserved for a future paper.

I wish to express my thanks for the interest Professor H. S. Taylor and Professor K. T. Compton have taken in this derivation.

[1] Wien's Displacement Law is proved by thermodynamic considerations and by experiment in contradistinction to Wien's Radiation Formula, which is only proved experimentally for small values of λ.

[2] Coblentz, W. W., *Bur. Standards Scient. Pap.*, No. 406 (1920).

CRITICAL POTENTIALS AND THE HEAT OF DISSOCIATION OF HYDROGEN AS DETERMINED FROM ITS ULTRA-VIOLET BAND SPECTRUM

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Lyman has found that under certain conditions of excitation the mixture of small proportions of hydrogen with argon causes the former to emit a series of eleven groups of lines, lying between $\lambda\lambda$ 1063 and 1670. Five of these groups he mentions in his book, "The Spectroscopy of the Extreme Ultra-Violet" (pp. 79–81), the others he discovered more recently. As shown in figure 1, which is a reproduction of one of Lyman's plates, these groups are at approximately equal intervals on a wave-length scale, except that one group is missing or very faint. Subsequently in studying and measuring Lyman's plates another group was found at the long wave-length end of the series. This group is very faint and has only two lines, whereas all the other groups have at least five or six each. If we number the groups 0, 1, 2, etc., beginning with the group of shortest wave-length and include the missing group, the faint group at the other end of the series is group 12 and the group that is missing or very faint is group 3. The zero group is partly obscured by one of the two extremely intense argon lines corresponding to its resonance potential of 11.6 volts. The other argon line is on the short wave-length side of this group.

The analysis made by the writer indicates that these groups constitute a one-dimensional series of bands, which have a common vibrational state in the upper level. We shall designate these bands the Lyman bands of hydrogen. They are degraded toward the red, and are apparently Q branches with perturbations near the origin. The one-dimensional character of the series seems to have the following explanation. Under the conditions of excitation the energy imparted to a hydrogen molecule is limited to 11.6 volts, the resonance potential of argon. This is just sufficient to raise the hydrogen molecule from the normal level with zero

vibrational energy to the first excited electronic level with zero vibrational energy. For the sake of convenience we shall introduce e_i for the total electronic quantum number in the sense in which this nomenclature is used in atomic spectra, and as usual designate by n and m the vibrational and nuclear rotational quantum numbers respectively. In this terminology we shall describe the process of excitation as raising the molecule from the state $e_i = 1, n = 0$ to the state $e_i = 2, n = 0$. The justification for this particular assignment of total electronic quantum numbers will be given later. The transitions made by the molecule in returning to lower levels are $e_i' = 2, n' = 0$ to $e_i'' = 1, n'' = 0$ to 12. According to the experimental results of Mulliken[1] and the theoretical results stated in a recent paper by Born and Jordan[2] half-integers should be used for the vibrational quantum numbers, but for the sake of convenience integers half a unit lower than the true vibrational quantum numbers are used in this discussion.

In the preceding explanation it has been found necessary to assume that the energy imparted to a hydrogen molecule is limited to 11.6 volts. Since these bands were excited by an electrical discharge such as occurs in an ordinary discharge tube, and not by a stream of electrons of uniform velocity, this limitation of the energy itself requires an explanation. There are two possible causes, both of which are presumably operative at the same time. One is the excitation of hydrogen molecules by collisions of the second kind with excited argon atoms having an energy of 11.6 volts. The other is that the presence of the argon limits the velocity of the free electrons to 11.6 volts, its lowest resonance potential, since electrons with greater velocity would suffer inelastic impact with argon atoms and lose their energy. This explanation would seem to be in accord with the experimentally observed fact that a large proportion of argon is required in the mixture in order to bring out this simple series of bands to the exclusion of the rest of the secondary spectrum in the extreme ultraviolet.

Designating by E the vibrational energy in the lower or normal level, the curve $\left(n'', \frac{1}{h}\frac{\Delta E}{\Delta n''}\right)$ for these bands indicates that $\frac{dE}{dn''}$, which is a measure of the vibrational frequency, becomes zero shortly beyond $n'' = 12$, so that at $n'' = 12$ the molecule is on the verge of dissociation, and apparently $n'' = 13$ cannot exist. Evidently the heat of dissociation, D, of the hydrogen molecule is equal to the vibrational energy for that value of n'' at which the frequency is zero. The data permit the computation of this quantity with a fair degree of accuracy. The computations yield 4.22 volts or 97,350 calories as the lower limit of the value of D, and 4.375 volts or 101,000 calories as the upper limit of D. Finally, using a power series representation of E as function of n'', it was found that

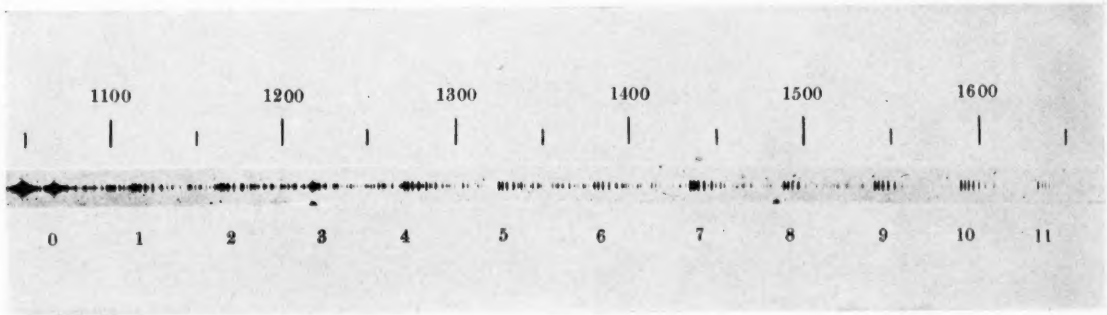
$$\frac{dE}{dn''} = 0 \text{ for } n'' = 12.86,$$

and $$D = E(12.86) = 4.27 \text{ volts} = 98{,}570 \text{ calories.}$$

This value of D, in the opinion of the writer, is not in error by more than 0.05 volt. It agrees with Isnardi's[3] result (as corrected by Wohl) that D lies between 96,765 and 102,257 calories. The value obtained by Langmuir[4] was 84,000 calories.

Other constants of the hydrogen molecule obtained from our data are: the lowest resonance potential = 11.61 volts; the frequency of infinitesimal vibration ω_0 in the normal state = 4260.cm.$^{-1}$, which is in fair agreement with the value 4880.cm.$^{-1}$, obtained by Kemble and Van Vleck[5] from specific heat data; and the value of $\chi\omega_0$ (coefficient of n^2) for the normal state = 112.5. cm.$^{-1}$.

The introduction of total electronic quantum numbers will now be discussed. Sommerfeld[6] refers to the ultra-violet band spectrum of hydrogen as holding the same relationship to the visible band spectrum that the Lyman series does to the Balmer series. Furthermore, the diagram due to Glitscher in the fourth and earlier editions of Sommerfeld's "Atombau und Spektrallinien" shows that even in the visible there is a close parallelism between the band spectrum and the Balmer series. This seems to indicate that the hydrogen molecule has approximately the same series of electronic energy levels as the hydrogen atom, and that it is possible to assign total electronic quantum numbers to the orbits of the radiating electron of the molecule in such a way that the difference in energy of two orbits of total quantum numbers e_t' and e_t'' will be nearly the same as the difference in energy of the orbits of the same total quantum numbers in the hydrogen atom. The energy of corresponding orbits is not the same, however. In the Glitscher diagram it will be observed that the peaks α, β, γ, δ are all on the short wave-length side of the corresponding lines of the Balmer series. This is in accord with the fact that the lowest resonance potential and the ionization potential of the molecule are each higher than the corresponding potentials of the atoms. All this shows that the electron in the nth quantum orbit in the molecule is more tightly bound than the electron in the nth quantum orbit in the atom. On the basis of this parallelism the lowest resonance potential of the molecule, which we found to be 11.61 volts, must correspond to 10.15 volts, the lowest resonance potential of the atom, and must be due to the electronic transition $e_t = 2$ to $e_t = 1$. This accounts for our previous assignment of total electronic quantum numbers. According to this scheme the ultra-violet band spectrum is due to the electronic transitions, $e_t' \geqslant 2$ to $e_t'' =$


1100
1200
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1, while the visible band spectrum is due to the electronic transitions, $e_i'' \geqslant 3$ to $e_i' = 2$.

On this theoretical basis an attempt will now be made to compute the higher resonance potentials and the ionization potential of the hydrogen molecule, using the Glitscher diagram. Let us assume that the peaks marked α, β, γ, δ on the Glitscher diagram represent electronic transitions corresponding to the electronic transitions of the hydrogen atom that cause the emission of the lines α, β, γ, δ, respectively, of the Balmer series. In other words these peaks represent the location of the (0, 0) bands for the corresponding electronic transitions. On this assumption by adding the voltage corresponding to each to these peaks to 11.61 volts one should obtain the higher resonance potentials of the molecule and by extrapolation the ionization potential. This is done below. The values of ν in table 1 are taken from the Glitscher diagram.

TABLE 1

PEAK	ν	$V = 1.2344 \times 10^{-4}$
α	16,650 cm.$^{-1}$	2.055 volts
β	21,850	2.697
γ	24,500	3.030
δ	25,900	3.197

TABLE 2

CRITICAL POTENTIALS

e_i	V	V(CALC)
2	11.61 volts	11.610 volts
3	13.665	13.631
4	14.31	14.323
5	14.64	14.639
6	14.81	14.810
∞	15.19	15.193

The critical potentials corresponding to jumps from the orbit $e_i = 1$ to higher orbits are given table 2. In the second column under V are given the values obtained by adding the voltages in table 1 to 11.61 volts. The value for $e_i = \infty$ was obtained by extrapolation. The theoretical discussion above as well as previous work of Fowler[7] and Sponer[8] suggested using a Rydberg formula to represent this series of values of V. The last column under V(calc) was computed from the Rydberg formula

$$V = 13.539 \left\{ \frac{1}{(1-\alpha)^2} - \frac{1}{(e_i-\alpha)^2} \right\}$$

with $\alpha = 0.0560$. It will be observed that this Rydberg formula gives a remarkably good representation of this series of values of V. Furthermore, even though the values obtained by the use of the Glitscher diagram be regarded as doubtful, still the value of 11.61 volts as the lowest reso-

nance potential would have yielded the series of values under V(calc). For we might have found α by solving the equation

$$11.61 = 13.539 \left\{ \frac{1}{(1-\alpha)^2} - \frac{1}{(2-\alpha)^2} \right\},$$

which, of course, gives $\alpha = 0.0560$, and therefore the previous values of V(calc).

While the azimuthal quantum numbers have not been considered in the preceding discussion, it seems natural to suppose that it will be necessary to introduce them in addition to the total electronic quantum numbers in order to define the electronic energy levels of the hydrogen molecule. If such is the case, there are two potentials corresponding to $e_i = 2$. The lower one of these, we know, is 11.61 volts. Consequently the other one must be higher. Now it may be that, in spite of the numerical agreement indicated in table 2, it is the higher of these two potentials that is related to the α, β, γ, δ peaks in the Glitscher diagram, and that instead of a Rydberg formula, we should use a Ritz formula. This would raise the value of the ionization potential, but probably not more than a few tenths of a volt.

Although the analysis of the secondary spectrum has not progressed to the point where one can make an accurate computation of the ionization potential of the molecule, nevertheless we may conclude that the spectroscopic data considered yield 15.19 volts as the probable value of that constant. It is possible that further investigation will raise this value several tenths of a volt, but it certainly does not seem as though the spectroscopic data would yield a value as high as the average obtained from electron impact experiments.

This apparent lack of agreement between the spectroscopic and the experimental values does not necessarily indicate that one or the other is wrong. For, according to recent work of Franck[9] and Birge and Sponer,[10] if an electronic transition causes a considerable change in the moment of inertia of a molecule, then it will not be possible in general to raise the electron from the lower to the higher level without exciting a certain minimum of vibrational energy. Where the excitation is by electron impact this energy must come from the impacting electron. It follows, therefore, that the minimum energy of an impacting electron required to produce a given electronic transition in the molecule will be equal to the difference in energy of the electronic levels in question plus the minimum vibrational energy excited by the electronic transition. This, of course, applies not only to the process of excitation, but to ionization also. It must, therefore, be expected that in these cases the observed critical potentials will be higher than the spectroscopic data would indicate.

In the case of hydrogen, the value of C, the coefficient of m^2, for the Lyman bands indicates that there is an extremely large change in the moment of inertia in going from the normal level to the first excited level. It is, therefore, highly probable that large changes in the moment of inertia occur when the electron is raised from the normal level to any higher level or when ionization occurs. If this assumption is correct, one would expect the observed critical potentials of the hydrogen molecule to be higher than those obtained from the spectroscopic data. This would account for the difference between the value of the ionization potential, 15.19 volts, obtained above and the value obtained from electron impact experiments.

If I_a is the ionization potential of the atom of an element, I_m that of its diatomic molecule, D the heat of dissociation of the neutral molecule, and D' that of the ionized molecule, it can easily be shown[11] that

$$I_a + D = I_m + D'$$

For hydrogen $I_a + D = 13.54 + 4.27 = 17.81$ volts.

If we take $I_m = 15.19$ volts,

$D' = 2.62$ volts.

Urey's[12] theoretical value of D' is 3.12 volts. Sommerfeld,[13] on the other hand, assumes that $D = I_m - I_a$, which automatically makes $D' = 0$. The data from positive ray analysis[14] shows that H^+ and H_3^+ appear at the same voltage as H_2^+. At low pressures ($H^+ + H_3^+$) forms only a small percentage of the total number of ions, but as the pressure increases the percentage of these ions increases also. This has been interpreted to indicate that the stability of H_2^+ is small or zero. In view of the fact that the energy with which the ions collide after their formation in the apparatus is unknown, this evidence must be considered inconclusive. Furthermore, according to the conclusions of the preceding paragraph, the ionized molecules produced in a postive ray apparatus will have considerable vibrational energy when formed. Therefore, unless they lose their energy in the meantime, it will require somewhat less than 2.62 volts to complete dissociation. Thus, if we assume the observed ionization potential to be 15.8 volts, the ionized molecules will have 0.6 volt of vibrational energy when formed and will require only 2.02 volts to complete dissociation.

The writer wishes to express his thanks to Prof. T. Lyman, who obtained the spectrum plates used in this investigation, and to Prof. E. C. Kemble and Dr. R. S. Mulliken, whose criticism and suggestions have been of great value.

Note added in proof: Since this paper was written, it has been found that the two lines supposed to constitute group 12 are due to impurities. With this change in the data, the lower limit of the heat of dissociation, D, of the hydrogen molecule becomes 4.10 volts or 94,600 calories, the upper limit 4.50 volts or 103,800 calories and the probable value of D 4.34 volts or 100,100 calories.

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[1] Mulliken, *Physic. Rev.*, **25**, 119, 1925.

[2] Born and Jordan, *Zeit. Physik.*, **34**, 858, 1925.

[3] Isnardi, *Zeit. Elektrochemie*, **21**, 405, 1915. Wohl (*Ibid.*, **30**, 49, 1924) corrected a slight error in computation in Isnardi's work, thereby raising his value about 5000 calories.

[4] Langmuir, *J. Amer. Chem. Soc.*, **37**, 417, 1915.

[5] Kemble and Van Vleck, *Physic. Rev.*, **21**, 653, 1923.

[6] Sommerfeld, *Atombau und Spektrallinien*, pp. 519–520.

[7] Fowler, *Proc. Roy. Soc.*, **A 91**, 208, 1915.

[8] Sponer, *Zeit. Physik.*, **34**, 622, 1925.

[9] Franck, *Trans. Faraday Society*, 1925.

[10] Birge and Sponer, unpublished work.

[11] Cf. Birge and Sponer, abstract 18, and Birge, abstract 19, March, 1926, meeting, Amer. Phys. Soc.

[12] Urey, *Physic. Rev.*, **27**, 216, 1926.

[13] Sommerfeld, *Atombau und Spektrallinien*, p. 518.

[14] Smyth, *Physic. Rev.*, **25**, 452, 1925; Hogness and Lunn, *Ibid.*, **26**, 44, 1925.

CURVED SPACE-TIME AND RADIATION

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1. Nearly two years ago, studying electrodynamics in curved space-time I found[1] that Maxwell's equations impose on space-time a restriction which can be formulated by saying that a certain vector q determined by the curvature field must be the gradient of a scalar function, or

$$\text{Rot } q = 0. \qquad (1)$$

It also was shown at that time[2] that the expression for the electromagnetic tensor involves the function φ, of which q is the gradient, under the sin and cos signs so that if, under separation of time and space at a certain point, the time component q_t of q does not change with time the electric and magnetic forces at that point are periodic functions of time, the period being $2\pi/q_t$; we have what would seem to be monochromatic light of frequency $q_t/2\pi$.

It is important to note that in these statements the term periodicity does not involve the assumption that the phenomenon occupies a time interval greater than the period; it is applicable to any interval of time however small.

2. During the last two years I made several attempts to find some simple

type of space-time in which q would satisfy equation (1). It appeared that for all types which answered our requirements in other respects not only Rot q vanishes but q vanishes itself. I have not published these results because radiation seemed to be absent in these spaces which, therefore, did not appear to correspond to physical reality.

3. On the other hand, experimental evidence accumulated during the last years seems to favor the point of view of the "extreme" quantum theory according to which light consists of *impulses*, a light impulse being analogous to the phenomenon of a moving material point, or electron, so that the regions of space-time where radiation is present are *one*-dimensional.

Electron paths we consider[3] as singular lines of the curvature field of space-time and there is no reason why we should not try to interpret radiation paths also as singular lines which differ in some respects from those representing matter.

4. In considering this transition from the conception of regions continuously filled with radiation to the view that radiation is concentrated in singular lines it is useful to keep in mind how the corresponding transition is made in the case of matter.

In the theory of potential we may begin by considering continuously distributed density connected with the field by Poisson's equation

$$\Delta_2\phi = 4\pi\rho. \tag{2}$$

We may then think of a process of condensation of matter around some points as the result of which density becomes, in the limit, zero everywhere outside of singular points in which the entire mass is concentrated. Equation (2) now becomes

$$\Delta_2\phi = 0 \tag{3}$$

and this holds in regions of regularity; in the singular points density is, of course, not defined in the same sense in which it was defined before in finite regions. We study matter, i.e., singularities by studying the residues of the field with respect to these singularities. If the residues are all equal, or are multiples of the same elementary quantity, as seems to be the case in Physics, what was previously called density, or mass, or electric charge of a body becomes a purely statistical conception.

This chain of reasoning applies with appropriate modifications also to matter in four-dimensional space-time, singularities being in this case *singular lines*.

5. We consider now in a similar way radiation. We begin with the case when

$$q \neq 0 \tag{2'}$$

in finite regions (sect. 1). We next imagine that these regions shrink in

all dimensions but one so that in the limit they become singular lines. Outside of these lines.

$$q = 0 \qquad (3')$$

but it is different from zero in points of the singular lines. Since, as remarked above, the property of periodicity does not depend on the size of the region in which q differs from zero we may regard this property as preserved in the limit and speak in the limiting case of frequency (from the point of view of a certain space passing through a point on such a line) as given by the time components of q (the time direction being normal to the space considered). *This point of view seems to reconcile the wave theory of light with the emission theory;* on the one hand the singular line which represents a ray of light is characterized by the vector q which is connected with the period of periodically changing electric and magnetic forces; on the other hand, it is one-dimensional so that the light impulse does not occupy time or space and is in this respect similar to a moving material point. Intensity is from this point of view a purely statistical conception like that of the mass of a body.

We shall not give here any expressions for q in terms of the curvature tensor; there are several such expressions and it is not clear at present which of them is the most suitable; we also shall not consider how the vector q can be obtained as a residue which we would expect to be the case judging by the analogy with the case of matter (sect. 4); there are several indications that it will be necessary here, as in the case of mass,[4] to introduce separation of space and time. We shall only notice that all the expressions for q involve the quantity ω introduced before,[5] and this leads us to the following considerations.

6. In studying antisymmetric tensors of rank two we found[6] that outside of the regular case with two invariable planes intersecting in a point there can exist, in a Minkowski bundle, also what we may call a *singular* antisymmetric tensor the two invariable planes of which intersect *in a line;* analytically this case is characterized by the vanishing of the number ω mentioned above, and from the point of view of separated space and time this means that the electric and magnetic vectors are perpendicular to each other and of equal length; this property is usually assumed to belong to the electromagnetic field in the region of electromagnetic waves (pure electromagnetic wave, according to H. Bateman). It has been pointed out why it is improbable that this may happen in finite regions and it was assumed that ω vanishes in points only; but it is easily seen that if we assume that ω may also vanish *along lines* or *curves* this would serve our purposes equally well and, of course, from our present point of view it is natural to assume that ω *vanishes along singular lines representing light.*

7. In the case of matter we know from experiment that the residues of

all singular lines of the same kind are exactly equal (equality of charges of all electrons[7]). It is natural to expect some such relation to be true also in the case of singular lines representing radiation; looking for a corresponding property of light we, of course, think of the so-called quantum of radiation. The thing which is here a universal constant is the quotient of frequency and energy. As stated above, frequency from the point of view of an observer belonging to a certain space is given by the time component, i.e., the *projection of the vector q* on the time direction perpendicular to the space considered. Energy, on the other hand, seems to be of the nature of the cosine of the angle between two time directions (on the "special relativity theory," if v is the relative velocity of two spaces, the cosine of the angle is $1/\sqrt{1-v^2}$ and the first two terms of the expansion of this expression are proportional, respectively, to the mass—which is energy—and to the kinetic energy of a particle at rest in one of the spaces as observed from the other). We see thus that some such relation as the so-called frequency relation could be expected in the case of singular lines representing light.

8. In conclusion I would like to point out the connection of the preceding considerations with the idea of a *vector quantum* which has been discussed by L. de Broglie[8] and F. W. Bubb.[9] The former seems to consider the quantum path as of the same nature as the path of a particle of matter and he concludes back to periodicity being involved in the motion of matter. From our point of view the singularities corresponding to light are essentially different from those corresponding to matter so that the radiation singularities can possess periodicity (and polarization) while the others do not.

It is also interesting to notice that Mecke[10] considers the quantum as what amounts to a residue corresponding to a contour integral whereas from our point of view we should expect to have a surface integral. The difference may be due to the fact that this author considers a quantum as essentially a scalar quantity, whereas q is a vector.

[1] These PROCEEDINGS, **10**, 1924 (124–27).
[2] *Ibid.* (294–98).
[3] *Trans. Amer. Math. Soc.*, **27**, 1925 (106–136), esp. sections 9 and 10.
[4] These PROCEEDINGS, February, 1926.
[5] Paper cited under 3, p. 115.
[6] *Ibid*, p. 114.
[7] *Ibid*, p. 134; this question will be discussed in a note to appear shortly.
[8] Thèse, Paris, 1924. Also *Journal de Physique*, **7**, 1926 (1–12).
[9] *Phil. Mag.*, **49**, 1925 (824–38).
[10] *Zs. Physik*, **21**, 1924 (26–37).

LINKAGE BETWEEN A FLOWER COLOR FACTOR AND SELF-STERILITY FACTORS

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Previous investigations concerning the nature and inheritance of self-sterility in plants have shown that the progeny of crosses between two self-sterile individuals fall into a number of definite sterility groups in which each member is sterile with all the other members of the same group, yet is fertile with members of different groups (Cardamine,[1] Linaria,[2] Veronica,[3] Antirrhinum,[4] Nicotiana[5,6,7,8]). Self-sterility and cross-sterility between members of the same sterility group are identical in nature. If pollen from a given plant is placed on the stigma of the same plant or of any other plant belonging to the same sterility group, pollen-tube growth is so slow that the flower ordinarily dies before two-thirds of the distance to the ovary has been traversed. In compatible matings, on the other hand, pollen-tube growth is rapid and fertilization easily accomplished. There is seldom any difficulty in distinguishing between these two cases, yet incompatible matings produce some seed if the pistils are pollinated in the young bud or if, for any reason, flowers persist for a few days longer than is usual.

In certain populations of Nicotiana plants the inheritance of three sterility groups has been investigated in detail.[8] It has been shown that these three groups—*X*, *Y* and *Z*—owe their existence to the action of three allelomorphic sterility factors—S_1, S_2 and S_3. Members of group *X* are S_1S_3, members of group *Y* are S_1S_2 and members of group *Z* are S_2S_3.

The effect of the sterility factors is such that the growth of pollen tubes carrying a given factor is inhibited in the styles of plants carrying this factor.[9] As a result, the flower usually dies before the ovary is reached. Pollen tubes carrying the same factor, however, grow rapidly enough down the styles of plants *lacking* that factor to accomplish fertilization. This interpretation is in agreement with the observation, already mentioned, that self-sterility and cross-sterility are identical in their nature. Thus an S_2S_3 plant, which produces only S_2 and S_3 pollen, is not only self-sterile, but also reciprocally cross-sterile with every other S_2S_3 plant.

A peculiarity in the inheritance of these sterility groups is that reciprocal crosses between members of different groups produce unlike progenies. When, for example, an S_2S_3 plant is pollinated by an S_1S_3 plant, only the S_1 pollen functions, the growth of the S_3 pollen-tubes being inhibited because the female carries S_3. The S_2 and S_3 ovules of the female fertilized by this S_1 pollen give progenies consisting of the groups S_1S_2 and S_1S_3

in approximately equal numbers. If the reciprocal cross is made ($S_1S_3 \times S_2S_3$) the S_1 and S_3 ovules are fertilized by S_2 pollen, the resulting progenies consisting of the groups S_1S_2 and S_2S_3 in approximately equal numbers.

Since reciprocal crosses give different results so far as the transmission of the sterility factors is concerned, it follows that any factor linked with the sterility factors must show corresponding differences. The purpose of the present paper is to give the results of a study of two flower color factors to determine their genetic relations with the sterility factors.

The plants used were members of populations of *Nicotiana Sanderae* (Hort.) = (*N. alata* Lk. and Otto × *N. forgetiana* Hort. (Sand.)), used by East and his co-workers for the determination of the inheritance of the sterility groups.[7,8] Reciprocal crosses between a number of plants were made in the greenhouse during the winter 1924–1925 and the resulting progenies were grown in field plots the following summer. Each plant was crossed with a sufficient number of test plants to determine its sterility group and the flower color of each plant was recorded.

Inheritance of Ivory Flower Color.—The first character to be considered is ivory (*ii*) so-called from the color of the homozygous recessive. The inner surface of the corolla lobes is ivory-colored, while the outer surface is tinged with magenta. The flower color of the heterozygous (*Ii*) and the homozygous dominants (*II*) is non-ivory, the shade of color depending upon the other factors present. The seeds of all three types are brown.

Table 1 shows the inheritance of ivory in three types of crosses. Backcrosses of plants heterozygous for ivory (*Ii*) on ivory (*ii*) give non-ivory and ivory plants in approximately equal numbers. Crosses between plants heterozygous for ivory (*Ii*) give progenies which approximate the expected 3:1 ratio. In both cases there is a small deficiency (about 6%) of the homozygous recessives (*ii*) which may or may not be significant. Progenies from crosses between heterozygous non-ivories (*Ii*) and homozygous non-ivory (*II*) contain no ivories.

The point of especial interest in table 1 is that reciprocal crosses give approximately the same results. Such differences as exist are small and fluctuate in both directions. Since linkage between "*i*" and the sterility allelomorphs must result in differences in the proportions of ivories in the progenies of reciprocal crosses, it is evident that the "*i*" factor and the sterility factors segregate independently of each other.

Inheritance of White Flower Color.—Plants possessing the basic color factor "*C*" show a purplish color at the base of the stem, color in the flowers and brown, rounded seeds. In plants which are recessive for this factor (*cc*) the stem is green, the flowers white and the seeds white and angular.

TABLE 1

INHERITANCE OF IVORY FLOWER COLOR

CROSS	FAMILY	ORIGIN	NON-IVORY	IVORY	PER CENT IVORY
Heterozygous	NA	N1 × 2R1C13	39	28	41.8
Non-Ivory *X*	RA	R1 × 2R1C13	11	12	52.2
Ivory	RD	R1 × 8R3R1	13	11	45.8
		Total	63	51	44.7
Heterozygous	NB	N1 × 7R1L1	43	9	17.3
Non-Ivory *X*	RB	R1 × 7R1L1	41	10	19.6
Heterozygous	NE	N1 × 23L1Q2	46	9	16.4
Non-Ivory	RF	R1 × 23L1Q2	37	7	15.9
		Total	167	35	17.3
Heterozygous	NC	N1 × 8L1Q2	36	0	0
Non-Ivory *X*	RC	R1 × 8L1Q2	54	0	0
Homozygous	ND	N1 × 22L1Q2	55	0	0
Non-Ivory	RE	R1 × 22L1Q2	52	0	0
		Total	197		
Ivory *X*	AN	Reciprocal of NA	33	27	45.0
Heterozygous	AR	Reciprocal of RA	16	12	42.9
Non-Ivory	DR	Reciprocal of RD	14	15	51.7
		Total	63	54	46.2
Heterozygous	BN	Reciprocal of NB	66	15	18.5
Non-Ivory *X*	BR	Reciprocal of RB	66	9	12.0
Heterozygous	EN	Reciprocal of NE	73	19	20.7
Non-Ivory	FR	Reciprocal of RF	59	25	29.8
		Total	264	68	20.5
Homozygous	CN	Reciprocal of NC	90	0	0
Non-Ivory *X*	CR	Reciprocal of RC	92	0	0
Heterozygous	DN	Reciprocal of ND	100	0	0
Non-Ivory	ER	Reciprocal of RE	91	0	0
		Total	373		

Under low temperatures the flowers of the homozygous recessive may show a pinkish tinge, but there is no difficulty in distinguishing between their color and the color of the heterozygous or homozygous dominants. The difference of seed color and shape is constant. White (*cc*) is epistatic to ivory (*ii*).

TABLE 2

INHERITANCE OF WHITE FLOWER COLOR

CROSS	FAMILY	ORIGIN	PROGENY COLORED	PROGENY WHITE	PER CENT WHITE	CROSS	FAMILY	ORIGIN	PROGENY COLORED	PROGENY WHITE	PER CENT COLORED	PER CENT WHITE	DIFFERENCE IN PER CENT WHITE IN RECIPROCAL CROSSES
Colored	LD	L1 × 7R1L1	54	53	49.5	White	DL	Reciprocal of LD	51	19	72.9	27.1	22.4
S_2S_3 *X*	LE	L1 × 8L1Q2	47	69	59.5	S_1S_3 *X*	EL	Reciprocal of LE	93	28	76.9	23.1	36.4
White	LH	L1 × 22L1Q2	24	31	56.2	Colored	HL	Reciprocal of LH	67	17	79.8	20.2	36.0
S_1S_3	LJ	L1 × 23L1Q2	50	45	47.2	S_2S_3	JL	Reciprocal of LJ	73	25	74.5	25.5	21.7
		Total	175	198	53.1			Total	284	89	76.1	23.9	29.2
Colored	LB	L1 × 1O2O1	48	48	50.0	White	BL	Reciprocal of LB	37	85	30.2	69.8	19.8
S_2S_3 *X*	LG	L1 × 28R1E2	22	22	50.0	S_1S_2 *X*	GL	Reciprocal of LG	13	39	25.0	75.0	25.0
White	LK	L1 × 28L1Q2	48	58	54.6	Colored	KL	Reciprocal of LK	19	77	19.8	80.2	25.6
S_1S_2						S_2S_3							
		Total	118	128	52.0			Total	69	201	25.5	74.5	22.5
Colored	NB	N1 × 7R1L1	52	58	52.7	White	BN	Reciprocal of NB	81	19	81.0	19.0	33.7
S_1S_2 *X*	NC	N1 × 8L1Q2	37	63	63.0	S_1S_3 *X*	CN	Reciprocal of NC	90	17	84.1	15.9	47.1
White	ND	N1 × 22L1Q2	55	59	51.6	Colored	DN	Reciprocal of ND	100	20	83.3	16.7	34.9
S_1S_3	NE	N1 × 23L1Q2	57	51	47.2	S_1S_2	EN	Reciprocal of NE	92	14	86.8	13.2	34.0
	RB	R1 × 7R1L1	52	49	48.3		BR	Reciprocal of RB	76	18	80.9	19.1	29.2
	RC	R1 × 8L1Q2	44	40	47.6		CR	Reciprocal of RC	92	9	91.1	8.9	38.7
	RE	R1 × 22L1Q2	52	51	49.5		ER	Reciprocal of RE	91	18	83.5	16.5	33.0
	RF	R1 × 23L1Q2	54	52	49.0		FR	Reciprocal of RF	89	15	85.6	14.4	34.6
	RG	R1 × 39R1L1	47	39	45.2		GR	Reciprocal of RG	67	20	77.0	23.0	22.2
		Total	450	462	50.5			Total	778	150	83.8	16.2	34.3

Three colored plants L1, R1 and N1, each having the constitution *Cc* were crossed reciprocally with a number of white plants (*cc*). The results of these crosses are shown in table 2. Here reciprocal crosses give quite different results. Whenever the heterozygous colored plants L1, R1 and N1 are used as female parents and the homozygous recessive whites are used as male parents—that is, when *C* and *c* ovules are fertilized by *c* pollen, the resulting progenies must consist of white and colored plants in equal numbers. The table shows that this result was obtained. In the total of 1531 plants, 51.5% were white and 48.5% colored (expected 50.0% ± 0.86%).

In the reciprocal crosses where the white plants (*cc*) were used as female parents and the heterozygous colored plants (*Cc*) were used as male parents, equality of colored and white plants can only result if equal numbers of "*C*" and "*c*" pollen tubes accomplish fertilization. It is clear from table 2 that the proportion of colored and white plants in these progenies is far from equal.

These differences in reciprocal crosses might be attributed to a specific effect of the "*C*" and "*c*" factors themselves on the rate of pollen tube growth, if it were not for the fact that in two groups of crosses there is a large excess and in the third an equally large deficiency of colored as compared with white plants. However, if there is linkage between the color factor and the sterility allelomorphs, the results obtained are consistent.

The first group of crosses in table 2 (right half) shows the results obtained when several white flowered plants of the sterility group S_1S_3 were pollinated by the colored L1 plant (S_2S_3). In this cross only the S_2 pollen functions. If the recessive color factor "*c*" is linked with the sterility factor S_2 the offspring should be white (*cc*), except for the cross-overs. If the dominant color factor "*C*" is linked with S_2 the offspring should be colored except for the cross-overs. More than three-fourths of the offspring were colored, indicating that the latter is the case. The constitution of the L1 plant is, therefore (S_2C S_3c).

This assumption is confirmed by the results of the second group of crosses reported in table 2. Here the same L1 plant is used as the male parent with several white plants of the sterility group S_1S_2. Since in these crosses the S_3c (non-cross-over) pollen and the S_3C (cross-over) pollen functions, the progeny should be predominantly white. Three-fourths of the offspring were white.

In the third group of crosses in the table, the two plants, R1 and N1, both having the constitution S_1S_2, were used as male parents on a number of white S_1S_3 plants. Here only the S_2 pollen functions. There is an excess of colored over white plants in the offspring of these crosses (83.8% colored instead of 50%). It follows that here again the S_2 factor is linked with the dominant color factor "*C*," the white flowered plants being pro-

duced by the cross-overs S_2c. The constitution of both the R1 and the N1 plant is, therefore, $S_1c.S_2C$.

Frequency of Crossing-Over between C and the Sterility Factors.—The frequency of crossing-over between the color factor and the sterility allelomorphs in microsporogenesis can be determined from the proportions of white and colored plants in the offspring (table 2, right half). As explained above the cross-overs consist of white flowered plants in the first and third groups and of colored plants in the second group of crosses. The total data are compiled separately for the three plants (table 4). The cross-over percentage in microsporogenesis for L1 is 24.5%, for R1 16.2% and for N1 13.6%.

TABLE 3

LINKAGE BETWEEN WHITE FLOWER COLOR AND THE STERILITY FACTORS IN MEGASPOROGENESIS

PLANT L1

FAMILY	NON-CROSS-OVERS COLORED S_1S_2	NON-CROSS-OVERS WHITE S_1S_3	TOTAL	CROSS-OVERS WHITE S_1S_2	CROSS-OVERS COLORED S_1S_3	TOTAL	PER CENT CROSS-OVERS
LB	39	36	75	5	5	10	11.7
LD	49	43	92	7	4	11	10.7
LE	32	47	79	20	12	32	28.8
LG	17	16	33	6	5	11	25.0
LH	19	21	40	7	4	11	21.6
LJ	35	31	66	4	12	16	19.5
LK	37	49	86	5	9	14	14.0
	228	243	471	54	51	105	18.2

PLANT R1

FAMILY	NON-CROSS-OVERS COLORED S_2S_3	NON-CROSS-OVERS WHITE S_1S_2	TOTAL	CROSS-OVERS WHITE S_2S_3	CROSS-OVERS COLORED S_1S_3	TOTAL	PER CENT CROSS-OVERS
RE	37	38	75	4	7	11	12.8
RF	42	34	76	9	4	13	14.6
RG	35	27	62	8	4	12	16.2
	114	99	213	21	15	36	14.5

For the determination of the crossing-over percentage in megasporogenesis both the color and sterility group for each plant in the progenies must be known. The data for the crosses in which L1 and R1 were used as female parents with several white plants are given in table 3.

In the first group of crosses tabulated in this table the L1 plant ($S_2C.S_3c$) was pollinated with pollen of white flowered $S_1c.S_3c$ and $S_1c.S_2c$ plants. In both cases only the S_1c pollen functions. The L1 plant forms four types of eggs: non-cross-overs S_2C and S_3c, and cross-overs S_2c and S_3C, all of which are fertilized by the S_1c male gametes. In the offspring the colored ($S_2C.S_1c$) and the white ($S_3c.S_1c$) plants are the non-cross-overs and the white ($S_2c.S_1c$) and the colored ($S_3C.S_1c$) plants are the cross-overs.

In the second group of crosses in table 3 the data for the cross between the R1 plant ($S_1c.S_2C$) and several white plants ($S_1c.S_3c$) are given. The non-cross-over eggs of R1—S_1c and S_2C—and the cross-over eggs—S_1C and S_2c—are fertilized by the S_3c male gametes of the white plants used. In the offspring, therefore, the white ($S_1c.S_3c$) plants and the colored ($S_2C.S_3c$) are the non-cross-overs, and the colored ($S_1C.S_3c$) and the white ($S_2c.S_3c$) plants are the cross-overs.

The total data as compiled in table 4 show 18.2% crossing-over for megasporogenesis in the L1 plant and 14.5% in the R1 plant.

TABLE 4

TOTAL CROSSING-OVER PERCENTAGES FOR WHITE FLOWER COLOR AND THE STERILITY FACTORS

NAME OF PLANT	MICROSPOROGENESIS TOTAL	MICROSPOROGENESIS PER CENT CROSS-OVERS	MICROSPOROGENESIS P. E.	MEGASPOROGENESIS TOTAL	MEGASPOROGENESIS PER CENT CROSS-OVERS	MEGASPOROGENESIS P. E.
L1	643	24.5	±1.15	576	18.2	±1.09
R1	495	16.2	±1.12	249	14.5	±1.51
N1	513	13.6	±1.03			

All data are in agreement with the assumption of linkage between the basic color factor "*C*" and the sterility allelomorphs "*S*." The average crossing-over value approximates 18%. The experiments are not sufficiently extensive to determine whether the differences in the per cent of crossing-over between individuals or between microsporogenesis and megasporogenesis are biologically significant.

Pollen-Tube Growth Factors in Other Plants.—Factors affecting the rate of pollen-tube growth have been postulated in several other cases to explain deviations from expected ratios. (Lychnis,[10] Rumex,[11] Oenothera,[12,13] Zea[14,15]). The situation in these cases apparently differs from that in Nicotiana only in that pollen-tube growth is less strongly inhibited, thus resulting in a deficiency of the class of segregates affected instead of its total absence.[16] It seems quite likely that factors affecting the rate of pollen-tube growth are not uncommon. Any linkage of visible characters with such pollen-tube growth factors must obviously result in disturbed ratios, the degree of disturbance depending upon the closeness of the linkage and the degree of inhibition of the pollen-tube growth. Where aberrant ratios which fail to explain themselves on other grounds are met with in genetical studies, the desirability of reciprocal crosses to test for differential pollen-tube growth is evident.

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ON THE OCCURRENCE IN THE HOUSE MOUSE OF A MENDELIZING STRUCTURAL DEFECT OF THE RETINA PRODUCING BLINDNESS[1]

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In a previous report[2] the occurrence of an hereditary eye defect in the domestic mouse was briefly described and certain suggestions were made as to its nature and the possible mode of its inheritance. It was described as a "retinal abnormality" (designated in the present paper as "rodless") characterized by the absence of the visual cells (rods) the external nuclear layer, and the external molecular layer. Subsequently it has been found that, in general, the external nuclear layer is not absent entirely but may be clearly demonstrated by differential staining to be reduced to a single row of nuclei which are packed against the internal nuclear layer thus eliminating the external molecular layer. The rods are entirely lacking, as stated. The remainder of the retina is normal in appearance and there are no signs of degenerative change or pathological condition of the tissues. (See Fig. 1.)

Identification.—Because of my inability to recognize this character by external observation, it has been necessary to section a retina from each animal in the families studied, except in the case of F_1 individuals from an outcross, in which the character is never known to occur, since it behaves as a complete recessive.

Function.—On account of difficulties encountered in studying rodent behavior, elaborate apparatus and a system of education were necessary before the functional status of such an eye could be determined. An attempt was made to build up association between "light" and "safety"

as well as "dark" and "electric shock" in animals of rodless inheritance and in controls of a normal strain.

As an example of these tests it may be stated that five control animals became educated well enough to choose lighted holes 180 times in 220 trials, in which they were offered a choice between one lighted and five unlighted holes. They thus chose the lighted hole 143 times oftener than chance alone would have taken them into a lighted hole, the probable error being only 3.71. This deviation is 38.6 times the probable error and shows clearly that the control animals were enabled by sight to choose the lighted hole.

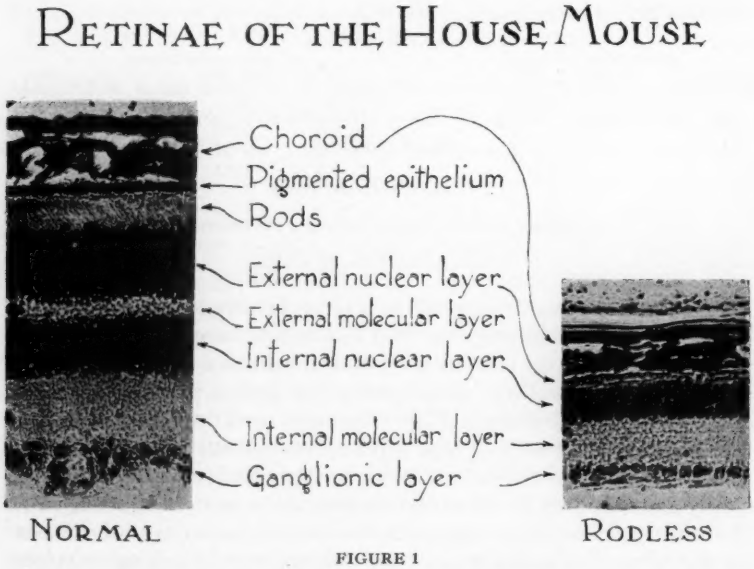

FIGURE 1

In a similar series of tests five rodless animals made 32 correct choices out of a possible 189. Since 32 is almost exactly one-sixth of 189, and in the experiments one in six of the holes was lighted, it is obvious that these animals entered the lighted hole purely by chance, their inability to see making it impossible for them to make a correct choice and thus avoid the ensuing electric shock, which normal seeing mice learned to avoid. The deviation from one-sixth is only 0.5, where the probable error, 3.37, allows for a much greater deviation through chance alone. Hence we are probably correct in assuming that this heritable structural abnormality carries with it total blindness as a functional consequence.

Sex Ratios.—Early tabulations of sex ratios are as follows:

Of homozygous rodless stock, bred *inter se*, 164 offspring were produced,

of which 86 were females and 78 were males. The deviation from the expected equality of the sexes is 4, the probable error being 4.3. Hence the sex ratio is entirely normal.

Of 125 F_2 rodless animals from crosses to normal strains, 62 were female and 63 male, again a normal sex distribution.

Back-cross rodless offspring totaling 244, consisted of 115 females and 129 males. The deviation from equality of the sexes is 7, or 1.3 times the probable error, 5.2, a result which is without statistical significance.

Accordingly everything indicates that the rodless character is neither sex-linked nor sex-limited.

Mendelizing Nature.—The early conclusion that the character is inherited as a Mendelian recessive was based on the data of a pedigree containing 31 individuals. No evidence to the contrary has been found in a population of over 2000 individuals examined since.

Viability.—In an F_2 population of 468 there were 351 normals expected and 343 found, 117 rodless expected and 125 found. The deviation is 1.2 times the probable error.

In back-cross animals totaling 511 there were expected 255.5 of each type, while 267 normals and 224 rodless were obtained. The deviation is 1.5 times the probable error, which is statistically insignificant, as are all previous deviations.

These ratios are evidently those of a Mendelizing recessive character bearing no lethal tendency.

Linkage.—Back-cross tests for linkage have been made involving albinism, dilution, brown, pink-eye, recessive spotting and agouti.

These give the following ratios of numbers of individuals in cross-over classes to numbers of individuals in non-cross-over classes:

For albinism, 26:20, dilution, 39:36; brown, 13:18; pink-eye, 17:19; recessive spotting, 26:20; agouti, 49:54. The deviations from equality are in no case significantly greater than the probable error.

A few data have been secured from tests for linkage with short-ears, kinky-tail, waltzing and dominant spotting. All results so far bespeak free segregation between rodless retina and all the characters enumerated. It is probable, therefore, that we are dealing with a character belonging to a new linkage group in mice.

Variations.—In the progeny of an outcross two other types have arisen, differing from the original abnormality in that one bears about 3 rows of nuclei in the external nuclear layer and the other about 6 rows. Strains are being established for study of the relationships involved, which suggest the possibility of modifying factors.

Embryology.—Studies upon the embryogenesis of the character have not progressed far enough to warrant the drawing of definite conclusions but the indications at present are of interest.

One is unable to recognize the character in sections of new-born mice, the development of the rodless eye not having deviated in a noticeable manner from the normal. At the thirteenth day, however, the difference is apparent. During this period in the normal animal the external and internal nuclear layers separate and the rods become differentiated. Histological examination of several series of retinae from this period would suggest that the rods are developed about the 6th day. This makes it probable that the rodless retina ceases growth about the 4th or 5th day of post-natal life.

Summary.—Summarily, it may be stated that this retinal anomaly, characterized by complete absence of rods and external molecular layer and reduction of the external nuclear layer, results in total blindness in the individual. It affects both sexes alike, breeds as a Mendelizing recessive, carries no lethal action and probably represents a new linkage group. It is possibly affected by modifying factors (as it varies quantitatively) and appears to result from a post-natal inhibition of differentiation in the nervous tissue ordinarily destined to produce the sensory receptor mechanism of the eye.

[1] Read before the Genetics Section of the American Society of Naturalists, December 28, 1925.

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VITAL STATISTICS OF THE NATIONAL ACADEMY OF SCIENCES. V. THE GROWTH OF THE ACADEMY

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The growth of the National Academy in total living membership can be ascertained by adding each year, to the living membership of the year before, the algebraic sum of the elections and deaths of the year in question. Thus, if at the end of the calendar year 1891 there were 97 members of the Academy, and during the calendar year 1892 there were 3 persons elected, and 7 persons died, the living strength of the Academy at the end of the calendar year 1892 was $97-4 = 93$. Proceeding in this manner throughout the history of the organization gives the living membership on December 31st of each calendar year. For the practical purpose of getting a picture of the manner in which the Academy has grown since its inaugura-

tion, 1863, this procedure is presumably as satisfactory as would be the determination of the membership at any other date in the year.

Table 1 has been prepared in this way. Resignations have been included with the deaths, and later reëlections of persons who have resigned have been included with new elections. In preparing this table an error was discovered in the printed list in the 1923–24 Report, which was overlooked when that document was used in the earlier papers in this series.[1] One person (W. L. Elkin) recorded in the Report as elected in 1917 was actually elected in 1915. It should be noted that table 1 and all the following discussion is concerned with Domestic Members only, and takes no account of Foreign Associates.

TABLE 1

THE GROWTH OF THE NATIONAL ACADEMY OF SCIENCES

YEAR	MEMBERSHIP AT END OF YEAR	ELECTIONS IN YEAR	DEATHS AND RESIGNATIONS IN YEAR	YEAR	MEMBERSHIP AT END OF YEAR	ELECTIONS IN YEAR	DEATHS AND RESIGNATIONS IN YEAR
1863	47	48	1	1895	89	3	3
1864	47	3	3	1896	86	2	5
1865	51	5	1	1897	84	4	6
1866	51	3	3	1898	82	0	2
1867	49	0	2	1899	85	5	2
1868	56	7	0	1900	86	4	3
1869	58	3	1	1901	86	5	5
1870	58	0	0	1902	86	5	5
1871	56	0	2	1903	88	5	3
1872	76	23	3	1904	90	4	2
1873	78	10	8	1905	94	5	1
1874	82	5	1	1906	96	3	1
1875	87	6	1	1907	99	4	1
1876	97	11	1	1908	105	9	3
1877	97	4	4	1909	112	10	3
1878	99	4	2	1910	113	7	6
1879	101	3	1	1911	118	9	4
1880	99	2	4	1912	123	8	3
1881	100	2	1	1913	130	10	3
1882	94	1	7	1914	133	10	7
1883	94	5	5	1915	140	11	4
1884	95	5	4	1916	146	9	3
1885	99	5	1	1917	158	15	3
1886	98	0	1	1918	168	15	5
1887	98	3	3	1919	177	15	6
1888	99	3	2	1920	187	15	5
1889	99	5	5	1921	199	15	3
1890	101	4	2	1922	206	13	6
1891	97	0	4	1923	215	15	6
1892	93	3	7	1924	223	15	7
1893	91	0	2	1925	230	12	5
1894	89	0	2				

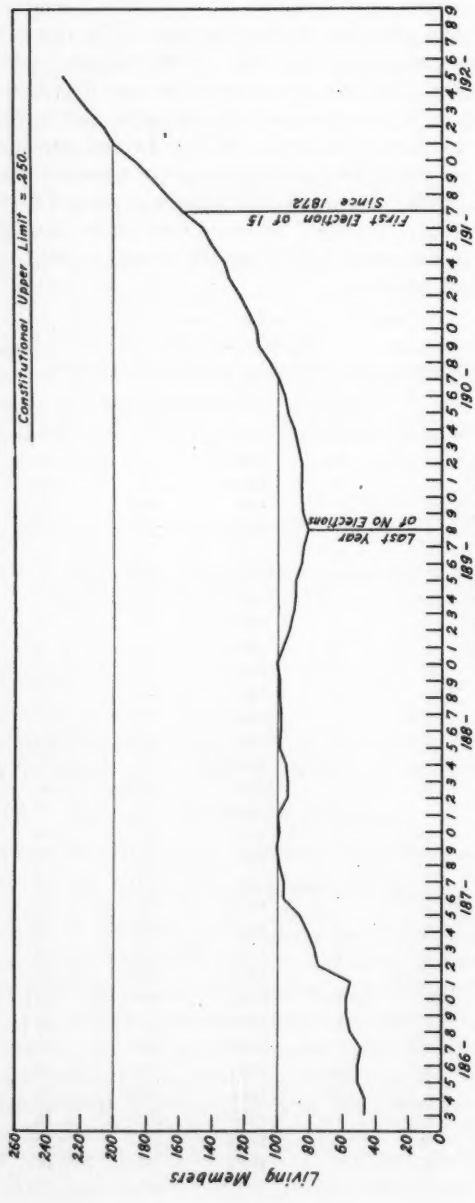

FIGURE 1

Showing the growth of the National Academy from 1863 to 1925, inclusive.

The data of table 1 are shown graphically in figure 1.

The data are interesting in several particulars. At the outstart the Academy took a period of 13 years in which to double its initial size, and only accomplished it in that period by electing 23 men in one year (1872). From 1876 to 1890 it oscillated only slightly about a membership of 100. From 1890 to 1898 the membership fell off. From 1898 to the present time the Academy has steadily increased in size, save for the two years 1901 and 1902, when elections just balanced deaths. From 1917, when the election of 15 persons a year began as a regular procedure, to 1924, the growth has been practically along a straight line. Between 1917 and 1925, inclusive, the average net gain in living membership per year was $9^1/_3$ persons. In the 14 years from 1907 to 1921 the Academy doubled in size a second time.

[1] *Proc. Nat. Acad. Sci.*, Vol. 11, pp. 752–768, 1925.

ON LAGUERRE'S SERIES. FIRST NOTE

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The object of the present note is to prove that the formal Laguerre's series of a certain class of functions are summable Abel. In the subsequent note the result is applied to the convergence theory of these series.

1. By a *Laguerre's series* we understand an expansion of the form

$$f(x) \sim \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x), \tag{1}$$

where

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} [x^{n+\alpha} e^{-x}], \tag{2}$$

α being real greater than -1, and the sign of equivalence is used to express the fact that the coefficients $a_n^{(\alpha)}$ are determined by the relations

$$a_n^{(\alpha)} = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \int_0^{\infty} e^{-t} t^{\alpha} L_n^{(\alpha)}(t) f(t)\, dt. \tag{3}$$

It is tacitly understood that the integrals exist in the sense of Lebesgue. The polynomials $L_n^{(\alpha)}(x)$ have the following properties of orthogonality:

$$\int_0^\infty e^{-t}\, t^\alpha\, L_m^{(\alpha)}(t)\, L_n^{(\alpha)}(t)\, dt = \begin{cases} 0 & , \text{ if } m \neq n, \\ \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}, & \text{if } m = n. \end{cases} \tag{4}$$

Further,

$$\int_0^\infty e^{-t}\, t^\alpha\, |f(t)|^2\, dt = \sum_{n=0}^\infty \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\, |a_n^{(\alpha)}|^2, \tag{5}$$

provided the integral exists in the sense of Lebesgue.

The special case $\alpha = 0$ has in many respects been more fully treated than the general one.[1] The corresponding functions $L_n^{(0)}(x) = L_n(x)$ are Laguerre's polynomials in the restricted sense. They were known to Abel, however.

2. The infinitary behavior of $L_n^{(\alpha)}(x)$ is very well known. The closest estimate can be obtained from the generating function

$$(1-z)^{-1-\alpha} \exp\left[-\frac{xz}{1-z}\right] = \sum_{n=0}^\infty L_n^{(\alpha)}(x)\, z^n, \tag{6}$$

where the series is convergent when $|z| < 1$, x being arbitrary finite. We shall only be concerned with real positive values of x, however. Using an estimate due to Fejér,[2] we find that

$$L_n^{(\alpha)}(z) = \frac{1}{\sqrt{\pi}}\, e^{\frac{x}{2}}\, x^{-\frac{1}{4}-\frac{\alpha}{2}}\, n^{\frac{\alpha}{2}-\frac{1}{4}} \cos\left[2\sqrt{nx} - \pi\left(\frac{1}{4}+\frac{\alpha}{2}\right)\right] + O\left(n^{\frac{\alpha}{2}-\frac{1}{2}}\right), \tag{7}$$

when $x > 0$ and α is real. From this we get for any fixed positive x

$$L_n^{(\alpha)}(x) = O\left(n^{\frac{\alpha}{2}-\frac{1}{4}}\right), \tag{8}$$

$$L_n^{(\alpha+1)}(x) = \sum_{m=0}^n L_m^{(\alpha)}(x) = O\left(n^{\frac{\alpha}{2}+\frac{1}{4}}\right). \tag{9}$$

The estimate is not true when $x = 0$, since $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$.

In virtue of (8) and (9) we can conclude that the series in (6) converges when $|z| \leqq 1$, provided $x > 0$ and $-1 < \alpha < -\frac{1}{2}$. When $-\frac{1}{2} \leqq \alpha < +\frac{1}{2}$, we still have convergence on the unit circle except at $z = +1$. For larger values of α we have divergence on the unit circle. The case $-1 < \alpha < -\frac{1}{2}$ is of special interest. Here we get convergent expansions of zero, namely

$$\sum_{n=0}^\infty L_n^{(\alpha)}(x) = 0, \; x > 0. \tag{10}$$

This fact implies that the representation of a function $f(x)$ by means of its Laguerre's series is not unique when $-1 < \alpha < -\frac{1}{2}$. In addition to the ordinary Fourier-Laguerre's series determined by (1) and (3), we have the expansions

$$\sum_{n=0}^{\infty} (a_n^{(\alpha)} + c) L_n^{(\alpha)}(x),$$

where c is arbitrary, which represent the function $f(x)$ whenever the Fourier-Laguerre's series gives a representation. The series (10) ceases to converge when $\alpha \geqq -\frac{1}{2}$; it is, however, summable (C, k) to the sum zero provided $k > \alpha + \frac{1}{2}$. Thus the series (10) is of some importance even to the summation theory of Laguerre's series. These expansions of zero seem hardly to have been noticed before.

3. We proceed to the summation theorem:

If a function $f(x)$ is such that the integral

$$\int_0^{\infty} e^{-at} t^{\alpha} \mid f(t) \mid dt \tag{11}$$

exists in the sense of Lebesgue for every $a > \frac{1}{2}$, then the series

$$F(x, r) = \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x) r^n, \tag{12}$$

where $\alpha > -1$, converges when $\mid r \mid < 1$, and

$$\lim_{r \longrightarrow +1} F(x, r) = f(x)$$

at every point of continuity of $f(x)$, $x \geqq 0$. In particular, if the series (12) is convergent when $r = +1$, $x = x_0$, $f(x)$ being continuous when $x = x_0$, then the sum of the series equals $f(x_0)$.

When $\alpha = 0$, this theorem reduces essentially to a theorem due to Wigert.[3] Mr. Wigert's proof for the special case can be carried over, almost word for word, to the general case. Owing to this fact and also to the considerable length of the proof, we shall merely enumerate the relations upon which the proof is based. We establish successively the following formulas:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)} a_n^{(\alpha)} r^n = (1-r)^{-1-\alpha} \int_0^{\infty} e^{-\frac{t}{1-r}} t^{\alpha} f(t) dt, \tag{13}$$

valid when $\mid r \mid < 1$. The convergence of (12) when $\mid r \mid < 1$ follows easily from this relation. In the subsequent formulas $J_{\alpha}(z)$ denotes Bessel's

function of the first kind and order α; all the powers that occur have their principal determination. In the application which we have in mind all the variables are real positive, but the formulas (14)–(18) hold for complex values as well. We have

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\frac{\alpha}{2}} \int_0^\infty e^{-u} u^{n+\frac{\alpha}{2}} J_\alpha(2\sqrt{xu})du, \tag{14}$$

$$\sum_{n=0}^\infty \frac{L_n^{(\alpha)}(x)}{\Gamma(n+\alpha+1)} t^n = e^t(tx)^{-\frac{\alpha}{2}} J_\alpha(2\sqrt{ru}), \tag{15}$$

$$\sum_{n=0}^\infty \frac{a_n^{(\alpha)}}{n!} r^n = e^r r^{-\frac{\alpha}{2}} \int_0^\infty e^{-u} u^{\frac{\alpha}{2}} J_\alpha(2\sqrt{ru})du, \tag{16}$$

$$\sum_{n=0}^\infty a_n^{(\alpha)} L_n^{(\alpha)}(x) r^n = e^x(rx)^{-\frac{\alpha}{2}} \int_0^\infty e^{-(1-r)t} J_\alpha(2\sqrt{xt}) \int_0^\infty e^{-u} u^{\frac{\alpha}{2}} J_\alpha(2\sqrt{rtu})\, f(u)du\, dt. \tag{17}$$

In the last formula, which is valid when $|r| < 1$, we have the right to interchange the order of the two integrations. Since $\alpha > -1$ and the real part of $(1-r)$ is positive, the integral with respect to t exists; its value can be found in any standard treatise on Bessel's functions.[4] Thus we obtain

$$F(x,r) = \frac{e^{-\frac{rx}{1-r}}(rx)^{-\frac{\alpha}{2}}}{1-r} \int_0^\infty e^{-\frac{u}{1-r}} u^{\frac{\alpha}{2}} e^{-\frac{\alpha\pi i}{2}} J_\alpha\left[2i\frac{\sqrt{rxu}}{1-r}\right] f(u)\, du. \tag{18}$$

A fairly simple discussion shows that this singular integral tends to $f(x)$ when $r \longrightarrow +1$, provided $f(x)$ is continuous for the particular value of x under consideration. At a point of discontinuity of the first kind we obtain as usual the mean value $\frac{1}{2}[f(x+0) + f(x-0)]$ as limit of $F(x, r)$.

This theorem is also of some importance for the theory of *Hermitian series*. In fact, putting $x = c\,u^2$ we have

$$L_n^{(-\frac{1}{2})}(c\,u^2) = C_n H_{2n}(u), \qquad uL_n^{(\frac{1}{2})}(c\,u^2) = C_n' H_{2n+1}(u),$$

where the values of c, C_n and C_n' depend upon which particular definition of the Hermitian polynomials we choose. By combining the summation theorems corresponding to $\alpha = -\frac{1}{2}$ and to $+\frac{1}{2}$, we obtain a summation theorem for Hermitian series which differs somewhat in form, though not in substance, from a summation theorem which the writer has given in a different connection.

[1] For the properties of the generalized Laguerre polynomials see G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. II, Julius Springer, Berlin, 1925, pp. 94, 293 and 294. The notation used by these writers agrees with the one used in the present note.

[2] See a paper by Perron, O., *Arch. Math., Leipzig* (3) **22**, 1914 (329–340) where reference is given to Fejér's paper as well as a considerable extension of his results.

[3] Wigert, S., *Ark. Matem., Stockholm*, **15** (25), 1921 (1–22).

[4] See, e.g., G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, 1922, p. 395.

ON LAGUERRE'S SERIES. SECOND NOTE

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The summation theorem stated in the preceding note can be utilized in the convergence theory of Laguerre's series. In fact, suppose that $f(x)$ is continuous, and that the integral

$$\int_0^\infty e^{-at}\, t^\alpha \,|\, f(t) \,|\, dt \qquad \text{(J)}$$

exists for every $a > \frac{1}{2}$. Then we know that the formal Fourier-Laguerre's series is summable Abel to $f(x)$ when $x \geqq 0$; thus the series will represent the function $f(x)$ whenever it is convergent. The expansion problem is consequently reduced to a simpler problem, namely that of finding sufficient conditions for the convergence of a given Laguerre's series. Some such conditions will be given in the following.

1. We begin with the following theorem:

I. *The series*

$$f(x) \sim \sum_{n=0}^{\infty} a_n^{(\alpha)}\, L_n^{(\alpha)}(x) \qquad (1)$$

converges when $x > 0$ *and represents* $f(x)$, *if* (i) $f(x)$ *is absolutely continuous on every finite interval* $0 < a < x < b$, (ii) $\lim_{x \to 0} x^{\alpha+1} f(x) = 0$, *and* (iii) *the integrals*

$$\int_0^\infty e^{-t}\, t^\alpha \,|\, f(t) \,|^2\, dt \quad \text{and} \quad \int_0^\infty e^{-t}\, t^\alpha \,|\, t f'(t) \,|^2\, dt \qquad (2)$$

exist. The convergence of (1) *is uniform on* $0 < a \leqq x \leqq b < +\infty$.

PROOF:[1] Consider the integral

$$\int_{x_1}^{x_2} e^{-t} L_n^{(\alpha)}(t) t^{\alpha+1} f'(t)\, dt = I(x_1, x_2)$$

where $0 < x_1 < x_2 < +\infty$. In view of (i) $f'(x)$ exists almost everywhere on $(0, +\infty)$ and is summable over (x_1, x_2). Hence $I(x_1, x_2)$ exists. Further, $f(x)$ is an indefinite integral of $f'(x)$ on (x_1, x_2) and it is permitted to integrate by parts. Thus we obtain

$$I(x_1, x_2) = [e^{-t} L_n^{(\alpha)}(t)\, t^{\alpha+1} f(t)]_{x_1}^{x_2} -$$

$$\int_{x_1}^{x_2} e^{-t} t^{\alpha} [(1 + \alpha - t) L_n^{(\alpha)}(t) + t L_n'^{(\alpha)}(t)] f(t)\, dt. \qquad (3)$$

Now let $x_1 \longrightarrow 0$, $x_2 \longrightarrow +\infty$. Then $I(x_1, x_2)$ tends to a finite limit and so does the integral on the right-hand side of (3). Denote for the moment the integrated part by $f_n(t)$. In view of (ii) $\lim_{t \longrightarrow 0} f_n(t) = 0$. Hence $\lim_{t \longrightarrow \infty} f_n(t)$ exists and equals zero, since $f_n(t)$ is integrable over $(0, +\infty)$.

Now it follows from formula (6) of the first note that

$$(1 + \alpha - x) L_n^{(\alpha)}(x) + x L_n'^{(\alpha)}(x) = (n+1) L_{n+1}^{(\alpha)}(x) - n L_n^{(\alpha)}(x).$$

Hence, denoting the Fourier-Laguerre coefficients of $f(x) + xf'(x)$ by A_n, we have

$$A_n = (n+1)[a_n^{(\alpha)} - a_{n+1}^{(\alpha)}].$$

In view of the existence of the integral

$$\int_0^{\infty} e^{-t} t^{\alpha} | f(t) + tf'(t) |^2\, dt,$$

it follows from formula (5) of the first note that the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} | A_n |^2 = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} (n+1)^2 | a_n^{(\alpha)} - a_{n+1}^{(\alpha)} |^2 \qquad (4)$$

is convergent. A simple calculation shows that

$$\sum_{n=1}^{\infty} | n^s a_n^{(\alpha)} - (n+1)^s a_{n+1}^{(\alpha)} | \qquad (5)$$

converges, when $s < \frac{\alpha}{2} + \frac{1}{2}$. In proving this we have to use (4) together with (5) in the first note as well as well-known properties of the Γ-function.

On the other hand, the Dirichlet's series

$$\sum_{n=1}^{\infty} \frac{L_n^{(\alpha)}(x)}{n^s} \tag{6}$$

converges when $s > \frac{\alpha}{2} + \frac{1}{4}$, in virtue of formula (9) of the first note, x being fixed positive. Further, the convergence is uniform with respect to x on any finite positive interval $0 < a \leqq x \leqq b$, if s is fixed $> \frac{\alpha}{2} + \frac{1}{4}$.

Now give s a fixed value, $\frac{\alpha}{2} + \frac{1}{4} < s < \frac{\alpha}{2} + \frac{1}{2}$, and consider the series

$$\sum_{n=1}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x) = \sum_{n=1}^{\infty} n^s a_n^{(\alpha)} \frac{L_n^{(\alpha)}(x)}{n^s} \tag{7}$$

For this particular value of s the series (5) and (6) are convergent, x being positive. Hence, (1) converges when $x > 0$ and uniformly with respect to x on $0 < a \leqq x \leqq b < +\infty$ in view of the uniform convergence of (6) with respect to x on the same interval. The series (1) may diverge or oscillate when $x = 0$, since the series $\sum_{n=0}^{\infty} \binom{n+\alpha}{n} a_n^{(\alpha)}$ is not necessarily convergent.

2. The expansion problem for Laguerre's series has been investigated previously in the special case $\alpha = 0$ by M^me^ V. Myller-Lebedeff[2] and in the general case by H. Weyl[3] who both base their analysis upon the theory of integral equations. The results of the latter writer are by far the more general ones. Expressed in terms of the polynomial series rather than the series of orthogonal functions and with due change of independent variable and parameter, the result of Weyl for the case $\alpha \geqq 0$ can be stated as follows:

The formal series (1) *represents* $f(x)$, *if* (i) $\lim_{x \to 0} x^{\alpha} f(x) = 0$, (ii) $\lim_{x \to \infty} e^{-x} x^{\alpha} f(x) = 0$, (iii) $G(x) = e^{-\frac{x}{2}} x^{\frac{\alpha-1}{2}} [(x-\alpha)f(x) - xf'(x)]$ *is continuous when* $x > 0$, *and* (iv) $\int_0^{\infty} G^2(x)dx$ *exists.*

The condition for the case $\alpha < 0$ is slightly different in form. Weyl remarks that the continuity of the derivative can be replaced by a less restrictive condition if Hellinger integrals are used.

3. To the preceding solution of the expansion problem we shall add a few conditions for the convergence of a Laguerre's series which involve restrictions of a different nature.

II. *If* $\lim n^{\frac{\alpha}{2} - \frac{1}{4}} a_n^{(\alpha)} = 0$, *and if* $F(x, r)$, *defined by formula* (12)

of the first note, is a holomorphic function of r at $r = +1$ when $x = x_0$, then $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x_0)$ *is convergent.*

The condition of a_n implies that $\lim a_n^{(\alpha)} L_n^{(\alpha)}(x_0) = 0$; hence II is a consequence of the theorem of Fatou. If the series in question is the Fourier-Laguerre series of a function $f(x)$ which is continuous when $x = x_0$ and such that the integral (J) exists for all values of $a > \frac{1}{2}$, then $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x_0) = f(x_0)$. A similar remark applies to the theorems III and IV below. The condition $\lim n^{\frac{\alpha}{2} - \frac{1}{4}} a_n^{(\alpha)} = 0$ is necessary in order that $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x)$ shall converge on an interval or on a set of positive measure. This fact is readily seen from formula (7) of the first note in virtue of a theorem on trigonometric series, due to Cantor and extended by several authors in various directions.

III. *If* $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x_0)$, $x_0 > 0$, *is summable Abel and if* $n^{\frac{\alpha}{2} + \frac{3}{4}} a_n^{(\alpha)}$ *is bounded, the series is actually convergent.*

Since $| a_n^{(\alpha)} L_n^{(\alpha)}(x) | < C/n$, the theorem is a consequence of a well-known Tauberian theorem due to Hardy and Littlewood.

IV. *The series* $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x)$ *converges when* $x > 0$, *if either* (i) $-1 < \alpha < -\frac{1}{2}$ *and* $\sum_{n=0}^{\infty} | a_n^{(\alpha)} - a_{n+1}^{(\alpha)} |$ *is convergent, or* (ii) $-1 < \alpha < +\frac{1}{2}$ *and* $\sum_{n=0}^{\infty} | a_n^{(\alpha)} + a_{n+1}^{(\alpha)} |$ *is convergent.*

The proof of this theorem is based upon §2 of the first note. When $-1 < \alpha < -\frac{1}{2}$, the series $\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)$ is convergent. The convergence of this series together with that of $\sum_{n=0}^{\infty} | a_n^{(\alpha)} - a_{n+1}^{(\alpha)} |$ implies the convergence of $\sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x)$. When $-1 < \alpha < +\frac{1}{2}$, the series $\sum_{n=0}^{\infty} (-1)^n L_n^{(\alpha)}(x)$ is convergent. The convergence of this series together with that of $\sum_{n=0}^{\infty} | a_n^{(\alpha)} + a_{n+1}^{(\alpha)} |$ implies the convergence of $\sum_{n=0}^{\infty} (-1)^n a_n^{(\alpha)} (-1)^n L_n^{(\alpha)}(x)$. So far we have tacitly assumed $x > 0$. In the second case we can, however, permit x to be zero, provided $\alpha < 0$. If $\alpha > 0$, we require some additional condition to ensure convergence when $x = 0$. If $\alpha = 0$, such a condition

is given by $\lim a_n^{(0)} = 0$; this condition together with the convergence of $\sum_{n=0}^{\infty} | a_n^{(0)} + a_{n+1}^{(0)} |$ implies the convergence of $\sum_{n=0}^{\infty} a_n^{(0)} = \sum_{n=0}^{\infty} a_n^{(0)} \; L_n^{(0)}(0)$.

[1] It is sufficient to prove the convergence of the series. It follows from (2) with the aid of Schwarz's inequality that the integral (J) exists for every $a > \frac{1}{2}$.

[2] *Math. Annalen*, **64**, 1907 (388–416).

[3] *Ibid*, **66**, 1909 (273–324).

GROUPS CONTAINING A RELATIVELY SMALL NUMBER OF SYLOW SUBGROUPS

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If p^m is the highest power of the prime number p which divides the order g of a given group G then G contains $1 + kp$, k being a positive integer or zero, subgroups of order p^m and its subgroups of this order are known as Sylow subgroups. The object of the present paper is to consider some relative properties of the subgroups of order p^m contained in G when their number is less than $2p^2 + 1$. In particular, we shall prove that when G contains more than one but less than $p^2 + 1$ such subgroups then it must involve an invariant subgroup of order p^{m-1} and when it contains more than one but less than $2p^2 + 1$ such subgroups then it must involve either an invariant subgroup of order p^{m-2} or an invariant subgroup order p^{m-1}. It is clear that the former of these theorems is useful only when m exceeds unity while the latter is useful only when m exceeds 2. Moreover, it is obvious that if in $g = p^m r$ and the number r is fixed, it is always possible to make m sufficiently large so that G involves an invariant subgroup of order p^α, $\alpha > 0$, for all larger values of m.

To prove the former of the theorems noted above it may first be observed than when G involves less than $p^2 + 1$ Sylow subgroups of order p^m then every pair of these subgroups must have p^{m-1} operators in common, for if two such subgroups have $p^{m-\alpha}$ operators in common then one of them has p^α conjugates under the other. Hence it may be assumed in the proof of this theorem that if K_1 and K_2 represent two subgroups of order p^m contained in G then their cross-cut K_0 is of order p^{m-1} and is invariant under K_1 and also under K_2. If K_0 were not invariant under G there would be a subgroup K_3 of order p^m contained in G which would not involve K_0. Each of the two distinct subgroups of order p^{m-1} which K_3

has in common with K_1 and K_2, respectively, would transform K_0 into itself. As these subgroups generate K_3 it results that K_0 is invariant under K_3 and hence it must be found in K_3. This proves that K_0 is the cross-cut of all the Sylow subgroups of order p^m contained in G, and hence the following theorem has been established: *If every pair of Sylow subgroups of order p^m contained in a group has a cross-cut of order p^{m-1} then all these subgroups have a common cross-cut of this order and this is an invariant subgroup of the group.* It results from this theorem, in particular, that the symmetric group of order 24 contains an invariant subgroup of order 4.

Suppose now that G involves at least two subgroups K_1 and K_2 of order p^m whose cross-cut is of order p^{m-2}. It results from this that K_1 transforms K_2 into p^2 distinct conjugates, and hence the number of the subgroups of order p^m found in G is at least $p^2 + 1$. If G contains more than $p^2 + 1$ such subgroups K_1 must have p^{m-1} operators in common with each of the others since it is assumed that G involves less than $2p^2 + 1$ subgroups of order p^m. We shall prove that the subgroup of order p^{m-2} which K_1 and K_2 have in common is invariant under G, and in what follows this subgroup will be denoted by K_0. To simplify a proof of this theorem it will first be assumed that G contains exactly $p^2 + 1$ subgroups of order p^m.

Since K_1 has only p^{m-2} operators in common with each of the remaining p^2 Sylow subgroups of order p^m it results that each pair of these subgroups has exactly p^{m-2} operators in common. Moreover, if K_0 is not invariant under K_1 each of its p-conjugates under K_1 transforms each of the others into itself since every non-invariant subgroup of a prime power group is transformed into itself by at least one of its other conjugates. Hence it results that if K_0 is non-invariant under K_1 each of the $p^2 + 1$ Sylow subgroups of order p^m contained in G has p relatively commutative subgroups of order p^{m-2} which generate a subgroup of order p^{m-1} and are its cross-cuts with the other Sylow subgroups of order p^m contained in G. If K_3 is such a Sylow subgroup which does not involve K_0 then the two cross-cuts of K_3 with K_1 and K_2, respectively, must be distinct and generate a group of order p^{m-1} which transforms K_0 into itself. Since this subgroup of order p^{m-1} is supposed not to contain K_0, it and K_0 generate a Sylow subgroup of order p^m which has more than p^{m-2} operators in common with K_1. As we arrived at a contradiction by assuming that K_3 does not involve K_0 it results that K_0 is the common cross-cut of all the Sylow subgroups of order p^m contained in G whenever G has exactly $p^2 + 1$ Sylow subgroups of this order.

It remains to consider the case when G contains more than $p^2 + 1$ but less than $2p^2 + 1$ Sylow subgroups of order p^m and some two of these subgroups K_1 and K_2 have only p^{m-2} operators in common. Those of these subgroups which have p^{m-1} operators in common with K_1 cannot have

a common cross-cut of this order since the number of these subgroups which involved this cross-cut would in that case be a divisor of the total number of subgroups of order p^m contained in G. This is impossible since the latter number exceeds the former by p^2. Hence we may let K_3 and K_4 represent two Sylow subgroups of order p^m which have two distinct cross-cuts of order p^{m-1} with K_1. The cross-cuts of K_3 and K_4 cannot be of order p^{m-1} since otherwise K_4 would involve two subgroups of order p^{m-1} which would transform the cross-cut of K_1 and K_3 into itself. This is impossible since every Sylow subgroup of order p^m includes every subgroup of order p^a which is invariant under it. The cross-cut of K_3 and K_4 must therefore be of order p^{m-2}, and hence this cross-cut is also the cross-cut of the two subgroups of order p^{m-1} which K_1 has in common with K_3 and K_4. It must therefore be invariant under K_1. We proceed to prove that it is also invariant under G.

Both K_3 and K_4 must have a cross-cut of order p^{m-1} with some of the p^2 conjugates of K_2 under K_1. Since these cross-cuts can involve only p^{m-2} operators of K_1 and since the operators of K_1 which appear in the p^2 conjugates of K_2 under K_1 are found in the same subgroup of index p in K_1 it results that these conjugates involve the invariant subgroup of order p^{m-2} which is found in K_1 and is the cross-cut of K_3 and K_4. Hence it results that all the Sylow subgroups of order p^m contained in G have a cross-cut of order p^{m-2}. That is, *if a group contains less than* $2p^2 + 1$ *Sylow subgroups of order* p^m *and if at least two of these subgroups have only* p^{m-2} *operators in common then the group involves an invariant subgroup of order* p^{m-2} *but no invariant subgroup of order* p^{m-1}.

Suppose that G contains $p^2 + lp + 1$, $l < p$, Sylow subgroups of order p^m and that at least two of these subgroups K_1 and K_2 have only p^{m-2} operators in common. We proceed to prove that excluding K_1 only p of the subgroups of order p^m can have the same cross-cut of order p^{m-1} in common with K_1, so that no two of the l different sets of p conjugate Sylow subgroups of order p^m under K_1 involve the same subgroup of order p^{m-1} contained in K_1. It was noted above that all these Sylow subgroups have a cross-cut of order p^{m-2} and that the p^2 which are conjugate under K_1 have no other operator in common with K_1. If more than p of the others besides K_1 had the same cross-cut of order p^{m-1} with K_1 each one of these would have a cross-cut of order p^{m-1} with at least p of the said p^2 conjugates under K_1 and no two of such sets of p such subgroups could involve a common subgroup, otherwise this common subgroup would involve two different subgroups of order p^{m-1} which would transform into itself the same subgroup of order p^{m-1} in K_1. Hence the common subgroup of order p^m would have p^{m-1} operators in common with K_1, which is contrary to the hypothesis. This proves the following theorem: *If a group contains* $p^2 + lp + 1$, $l < p$, *Sylow subgroups of order* p^m, *and at least two*

of these subgroups have a cross-cut of order p^{m-2}, then each of them contains l subgroups of index p which have a common cross-cut of order p^{m-2} and are separately invariant under exactly p subgroups of order p^m.

It will now be proved that the only positive integral values which the l in the preceding theorem can have is 2. It is obvious that l cannot be unity since those subgroups of order p^m which have a cross-cut of order p^{m-1} with K_1 cannot have a common cross-cut of this order, as was noted above. If l were larger than 2 then at least $2p$ of the subgroups of index p in one set of p conjugate subgroups of order p^m under K_1 would appear in the p^2 conjugates of K_2 under K_1. As each of these subgroups would be a cross-cut of p of these p^2 conjugates one of these conjugates would involve at least two subgroups of order p^{m-1} found in the given set of p subgroups of order p^m which are conjugate under K_1. This is impossible since these two subgroups transform a given subgroup of order p^{m-1} in K_1 into itself and they also generate a group of order p^m. Hence the following theorem has been established. *If a group contains $p^2 + lp + 1$, $l < p$, Sylow subgroups of order p^m then it contains an invariant subgroup of order p^{m-1} except possibly when $l = 0$ or when $l = 2$. If in either of these two exceptional cases it does not involve an invariant subgroup of order p^{m-1} it must then involve an invariant subgroup of order p^{m-2}.*

By means of the preceding theorem it is easy to determine the number of operators whose orders are powers of p in a group G which involves less than $2p^2 + 1$ Sylow subgroups of order p^m except in the case when the number of these subgroups is $p^2 + 2p + 1$, $p > 2$, and when at least two of them have a cross-cut of order p^{m-2}. In this special case all the subgroups of order p^m contained in G have a common cross-cut of order p^{m-2} and the quotient group of each of these subgroups with respect to this common cross-cut is the non-cyclic group of order p^2. Hence each of them involves $p + 1$ subgroups of order p^{m-1} which have no operator in common besides the said common cross-cut of order p^{m-2}. Two of these $p + 1$ subgroups appear in $p + 1$ Sylow subgroups of order p^m while the remaining $p-1$ appear in only one such subgroup. Hence the total number of operators of G whose orders are a power of p is

$$p^{m-2} + (p^{m-1}-p^{m-2})[(p-1)(p^2 + 2p + 1) + 2(p + 1)] = p^{m+2}.$$

Since the number of operators whose orders are powers of p in a group which involves $p^2 + p + 1$ Sylow subgroups of order p^m is also p^{m+2} and this is the number of such operators in a group which involves $p^2 + 1$ such Sylow subgroups provided two of these subgroups have a cross-cut of order p^{m-2} it results that groups in which the number of the Sylow subgroups of order p^m is of any one of the following three forms

$$p^2 + 1,\ p^2 + p + 1,\ p^2 + 2p + 1$$

may involve the same number of operators whose orders are powers of p notwithstanding the difference in the number of such Sylow subgroups. In all the other groups which involve less than $2p^2 + 1$ Sylow subgroups of order p^m the number of operators whose orders are powers of p obviously increases with the increase in the number of such Sylow subgroups.

It was noted above that if the order of G is written in the form $p^m r$, where r is a fixed number prime to p, then m may be assumed to be sufficiently large so that G involves an invariant subgroup of order p^α, $\alpha > 0$. Since the number of the subgroups of order p^m contained in such a group must be a divisor of r it results from this that if the number of the Sylow subgroups is fixed it is always possible to assume that the order of such a subgroup is sufficiently large so that every group which involves such a number of these subgroups will involve an invariant subgroup, larger than the identity, composed of their cross-cut. Such an invariant subgroup certainly exists when m exceeds the index of the highest power of p contained in $r!$. From the known properties of the group of transformations of Sylow subgroups it results directly that such an invariant subgroup is usually present when m has a much smaller value.[1] In particular, it is easily seen that this group of transformations can never be more than triply transitive except in the special case when there are four Sylow subgroups whose order is a power of 3. Even in this case the group of transformations of these subgroups cannot be more than triply transitive if the symmetric group of degree n is said to be $(n-1)$-fold transitive, as is sometimes done. A proof of this theorem results almost immediately from the fact that the subgroup composed of all the substitutions which omit one letter of such a group of transformations involves an invariant Sylow subgroup whose order is p^α, $\alpha > 0$, if the order of the transformed Sylow subgroups is p^m. It may be added that a complete set of n conjugate subgroups of a group G may sometimes be transformed under G according to the symmetric group of degree n even if G is generated by two of these conjugates as is the case as regards the maximal subgroups of degree $n-1$ in the symmetric group of degree n.

[1] Miller, G. A,, "Groups of Transformations of Sylow Subgroups," *Amer. Jour. Math.*, **32**, 1910, p. 299.